Lecture 12
Einstein´s Field Equations

The Cosmological Principle

The Olber's Paradox

Space-Time Metric

Geodesics and Motion

Curvature

Energy-Momentum Tensor

Einstein´s Field Equations

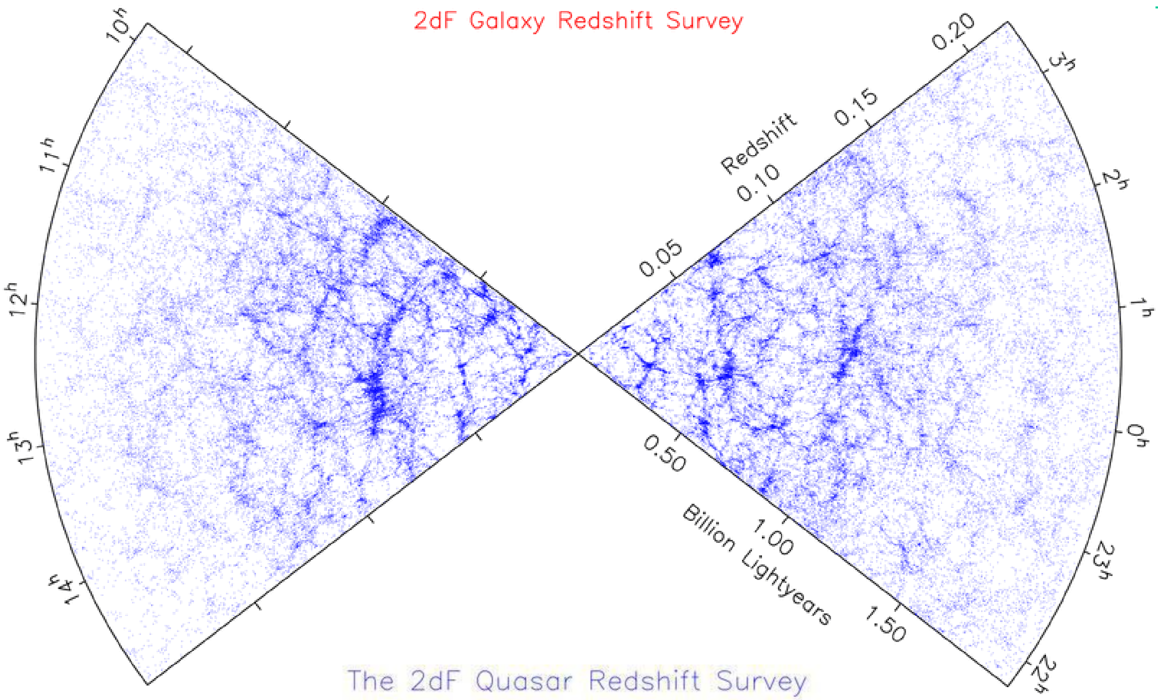
⇨ The Cosmological Principle

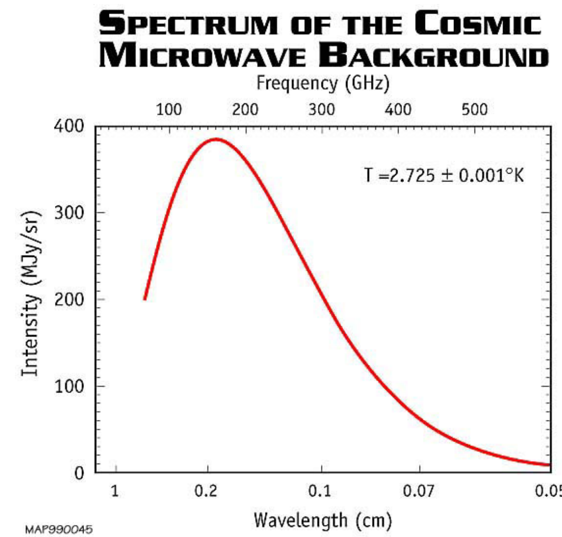
- ✓ Rudnicki [1995, *The Cosmological Principles*] summarized various forms of cosmological principles, in modern-day language, which were stated over different periods in human history based on philosophical considerations rather than on fundamental physical laws
  - **Ancient Indian**
    The Universe is infinite in space and time and is infinitely heterogeneous
  - **Ancient Greek**
    Our Earth is the natural center of the Universe
  - **Copernican**
    The Universe, as observed from any planet, looks much the same
  - **Modern**
    The Universe is (roughly) homogeneous and isotropic

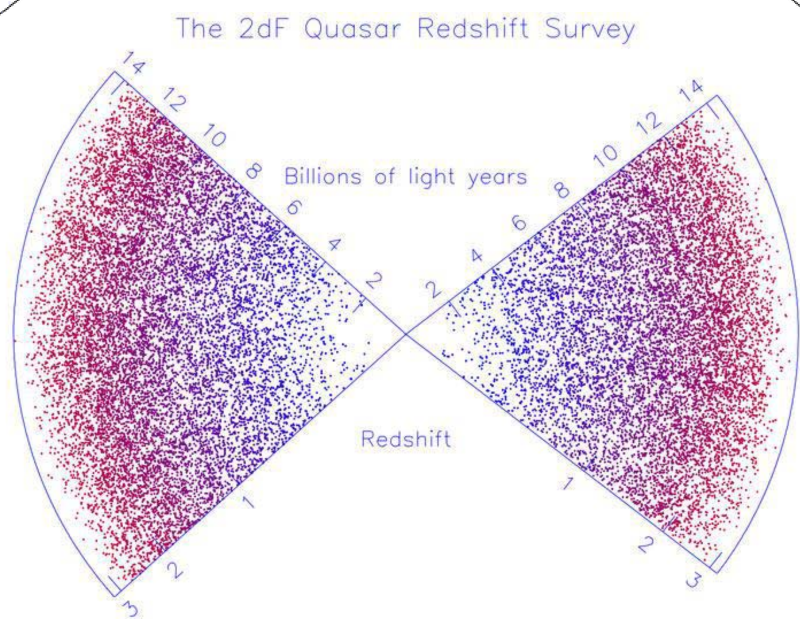
- ✓ The Universe is ***homogeneous*** if all (possible) commoving observers would observe **identical properties** for it (all space positions are equivalent)
- ✓ The Universe is said to be ***isotropic*** if for every commoving observer there is **no preferred direction** (properties of the Universe appear to be the same in all possible directions)
- ✓ **An isotropic Universe is always homogeneous, but the converse is not true**

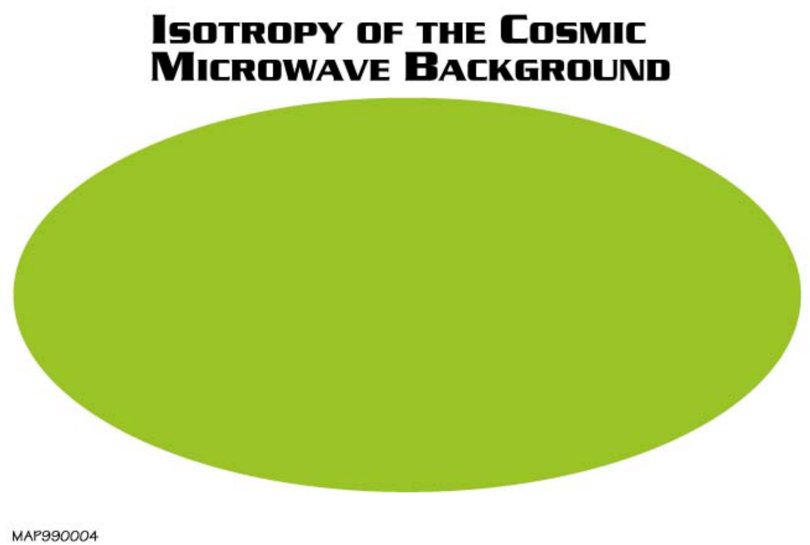
- Example of a homogeneous and isotropic "Universe": the 2D **surface of a sphere**
- Example of a homogeneous but not isotropic "Universe": the 2D **surface of an infinite circular cylinder**

⇨ The Cosmological Principle









⇨ The Olber's Paradox

- ✓ This paradox was discovered by the Greeks and rediscovered several times since
- ✓ It tells us that one of the most important observations in Cosmology is that it gets **dark at night**!

- ✓ Let's begin by making the following "reasonable" **assumptions**:
  1) the known **laws of physics**, which we have deduced locally, **are valid throughout** the Universe (Cosmological Principle)
  2) the mean density $\mathbf{n_s}$ and luminosity $\mathbf{L_s}$ of stars (or galaxies) is **independent of position** (also Cosmological Principle)
  3) the same quantities are **independent of time** (static Universe)
  4) there is **no large scale systematic motion** of stars (or galaxies) in the Universe
  5) **radiation is not absorbed**

- ✓ Now let **<u>** be the mean radiation energy density in the Universe:

$$\langle u \rangle = \int dV\, n_s \frac{L_s}{4\pi r^2 c} = \frac{n_s L_s}{4\pi c} \int 4\pi\, r^2\, dr \frac{1}{r^2} = \frac{n_s L_s}{c} \int dr = \infty$$

- ✓ **The night sky is clearly not that bright!**

⇨ The Olber's Paradox

- ✓ Including absorption by dust (**5**) would not help much: because of assumption 3, dust must eventually come into equilibrium with the stars, and it would then radiate as much as it receives. This would predict a night sky as bright as the surface of a typical star (T ~ 5000 K, but we know that the universe has T ~ 3 K)
- ✓ Relaxing assumption **4** can avoid the paradox: one only needs $v_{radial} \sim c$ at large distances [expansion of the Universe is consistent with cosmological principle (1 and 2) if it is uniform]
- ✓ Relaxing assumption **3** can also avoid the paradox: dust may not be hot yet, or stars may not have shined long

- ✓ Thus, Olber´s paradox tells us that the Universe is either **young**, or it is **expanding**, **or both**

⇨ Space-Time Metric

- ✓ *I. Newton* – flat ***Euclidean*** space
- ✓ *G.F.B. Riemann* – Euclidean space is just a particular choice suited to flat space, but not necessarily correct in the space we inhabit
- ✓ *E. Mach* – we should abandon the concept of absolute space because it is unobservable
- ✓ *A. Einstein* – replaced the flat Euclidean 3D space with a curved ***Minkowskian*** 4D space-time (in which physical quantities are described by invariants)

- Making use of the **tensor notation**, one can write **metric equations** quite generally as:

$$ds^2 \equiv \Sigma_0^3 \, g_{\mu\nu} \, dx^\mu \, dx^\nu = g_{\mu\nu} \, dx^\mu \, dx^\nu$$

where:

- **repeated indices** (in a product) mean **summation** (Einstein´s notation)
- **Greek indices** (**α**, **β**, **γ**, **μ**, **ν**, **σ**, etc) are used to represent all the four components of space-time, conventionally **0**, **1**, **2** and **3**, meaning **t**, **x**, **y** and **z**
- when only space coordinates are to be represented, **Latin indices** (**i**, **j**, **k**, **l**, etc) are used (**1**, **2**, **3**)
- **upper** and lower **indices** are distinct: the former is associated with vectors and the latter with 1-form
- the metric can be used to raise or lower indices on tensors:

$$g^{\mu\nu} = g^{\mu\alpha} \, g^{\nu\beta} \, g_{\alpha\beta}$$

$$g^{\alpha\nu} \, g_{\beta\nu} = \delta^\alpha_{\ \beta} \qquad \rightarrow \text{delta de Kronecker: } \begin{cases} 1 & (\alpha=\beta) \\ 0 & (\alpha\neq\beta) \end{cases}$$

⇨ Space-Time Metric

✓ *Euclidean* space (**Cartesian** coordinates)

$$ds^2 = dx^2 + dy^2 + dz^2 \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

✓ *Euclidean* space (**spherical** coordinates)

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

✓ ***Minkowski*** space-time (special relativity standard)

$$ds^2 = c^2\, d\tau^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2 \qquad g^{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

τ is called "***proper time***" (an **invariant** under Lorentz transf.)

✓ ***Robertson-Walker*** (R-W, 1934) space-time (cosmological standard)

$$ds^2 = c^2\, dt^2 - a^2(t) \left(\frac{dr^2}{1 - k\, r^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \right)$$

$$g^{\mu\nu} = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & -a^2/(1-kr^2) & 0 & 0 \\ 0 & 0 & -a^2 r^2 & 0 \\ 0 & 0 & 0 & -a^2 r^2 \sin^2\theta \end{pmatrix}$$

r, **θ** and **φ** are ***co-movel*** coordinates
(physical coordinate / *a*)

⇨ Geodesics and Motion

- ✓ ***Geodesic*** is a path that represents the **shortest distance** between two points (in *Euclidean* space it is a straight line, but not in General Relativity where matter-energy curve the space-time)

- ✓ The **geodesic equation**, or relativistic equation of motion, in an arbitrary gravitational field and an arbitrary coordinate system, can be represented by

$$\frac{Dx}{D\tau} \equiv \frac{d^2}{d\tau^2} x^\mu + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

where

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right)$$

are the **affine connections** or ***Christoffel*** **symbols**

⇨ Curvature

- ✓ **Curvature** is a quantity that characterizes the local deviation of the geometry from flatness
- ✓ since it is an invariant quantity, the curvature **does not depend on the coordinate system**

- ✓ The curvature of a generic space-time may be conveniently defined in terms of the ***Riemann* tensor**

$$R^{\mu}{}_{\alpha\beta\gamma} \equiv \frac{\partial \Gamma^{\mu}{}_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial \Gamma^{\mu}{}_{\alpha\beta}}{\partial x^{\gamma}} + \Gamma^{\mu}{}_{\sigma\beta}\, \Gamma^{\sigma}{}_{\gamma\alpha} - \Gamma^{\mu}{}_{\sigma\gamma}\, \Gamma^{\sigma}{}_{\beta\alpha}$$

- ✓ the antisymmetry property of the *Riemann* tensor shows that there are only 2 tensors that can be formed by **contracting** it: the ***Ricci* tensor**

$$R_{\alpha\beta} = R^{\mu}{}_{\alpha\beta\mu}$$

and the **scalar curvature**

$$\mathcal{R} = R^{\mu}{}_{\mu} = g^{\mu\nu}\, R_{\mu\nu}$$

- ✓ The ***Einstein* tensor** is a combination of the **Ricci tensor**, the **scalar curvature** and the **metric**

$$G^{\mu\nu} \equiv R^{\mu\nu} - \tfrac{1}{2}\, g^{\mu\nu}\, \mathcal{R}$$

and is unique as a tensor linear in second derivatives of the metric.

⇨ Energy-Momentum Tensor

- **All forms of energy produce gravity**
- The energy-momentum or stress tensor may has the following components:
  - T_{00} – energy density $\boldsymbol{\rho c^2}$, which includes the rest mass, the internal and the kinetic energies
  - T_{ii} – **pressure** components in the *i* direction, or the momentum components per unit area
  - T_{ij} – shear of the pressure component p^i in the j direction
  - cT_{0i} – energy flow per unit area in the *i* direction
  - cT_{i0} – momentum densities in the *i* direction

- a commoving observer, in a R-W spacetime, who follows the motion of the fluid, sees no time-space or space-time component
- the cosmological principle allows to neglect the anisotropic non diagonal space-space components
- thus, the energy-momentum tensor can be cast into purely diagonal form (*Weyl*'s Postulate)

$$T_{\mu\mu} = (P + \rho c^2)\,\frac{u_\mu u_\mu}{c^2} - P g_{\mu\mu}$$

- The **conservation** of energy and momentum can be written

$$\frac{DT^{\mu\nu}}{Dx_\nu} = 0$$

⇨ Einstein's Field Equations

- ✓ The **General Relativity** is a theory of **Gravitation**
- ✓ A. Einstein worked hard on searching for a "Law of Gravitation" in the framework of Relativity, combining the **principle of equivalence*** with the requirement of **general covariance#**
- ✓ since the basic idea is that gravity is not a force but a **bending of space-time** due to the **presence of matter and energy**, this law has to relate the geometry of space-time, the **metric** and **curvature**, to the distribution of **matter-energy**

$$\mathbf{G}^{\mu\nu} \equiv \mathbf{R}^{\mu\nu} - \tfrac{1}{2}\,\mathbf{g}^{\mu\nu}\,\mathcal{R} = -\,(8\pi G/c^4)\,\mathbf{T}^{\mu\nu} - \Lambda\mathbf{g}^{\mu\nu}$$

- ✓ **Λ** was included *ad hoc* on the equations by Einstein in order to produce a static Universe. Despite the latter repudiation by its creator, it has refused to die and has assumed a central role in Modern Cosmology. In the form in which it is written above, it represents the **curvature of the empty space**

* In any gravitational field, a freely falling observer will experience no gravitational effect, that is, an uniformly accelerated frame of reference mimics a gravitational acceleration.

The formulation of the laws of Physics in terms of quantities which remain invariant under an arbitrary acceleration is called general covariance.

Further readings

Books:

- S.Weinberg, 1972; *Gravitation and Cosmology*, Wiley Press, New York
- E.W. Kolb & M.S. Turner, 1990; *The Early Universe*, Addison-Wesley Pub. Co., California
- P.J.E. Peebles, 1993; *Physical Cosmology*, Princeton Univ. Press, Princeton
- M. Roos, 1999; *Introduction to Cosmology*, Wiley Press
- J.A. Peacock, 2000; *Cosmology Physics*, Cambridge Univ. Press
- S. Dodelson, 2003; *Modern Cosmology*, Academic Press
- M.H. Jones & R.J.A. Lambourne, 2003; *Galaxies and Cosmology*; Cambridge Univ. Press