Lecture 13
Friedmann Model

FRW Model for the Einstein Equations

First Solutions

- Einstein (Static Universe)
- de Sitter (Empty Universe) and H(t)

Steady-State Solution (Continuous Creation of Matter)

Friedmann-Lemaître Solution & Expansion

- The possible solutions for an homogeneous Universe
- First Law of Thermodynamics
- Equation of State

⇨ FRW Model for the Einstein's Field Equations

✓ The **Einstein's Field Equations** are of the following

$$G^{\mu\nu} \equiv R^{\mu\nu} - \tfrac{1}{2}\, g^{\mu\nu}\, \mathcal{R} = -\, 8\pi G\, T^{\mu\nu} - \Lambda g^{\mu\nu}$$

where $\mathbf{g^{\mu\nu}}$ is the space-time metric, $\boldsymbol{\Lambda}$ is the cosmological constant,

$$R^{\mu\nu} \equiv \frac{\partial \Gamma^{\alpha}{}_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\alpha}{}_{\mu\nu}}{\partial x^{\alpha}} + \Gamma^{\alpha}{}_{\sigma\nu}\, \Gamma^{\sigma}{}_{\alpha\mu} - \Gamma^{\alpha}{}_{\sigma\alpha}\, \Gamma^{\sigma}{}_{\nu\mu}$$

$$\mathcal{R} = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu} \qquad\qquad T_{\mu\mu} = (P + \rho c^2)\, \frac{u_{\mu} u_{\mu}}{c^2} - P g_{\mu\mu}$$

✓ For applying these equations to the **whole Universe**, we first need to assume the **Cosmological Principle**, which means that we need to replace the **R-W metric** on them. Since the R-W metric is **diagonal**, we only need to consider the terms "μμ". The calculation of the diagonal terms show that the spatial ones are redundant, that is, in fact we only need the terms "**00**" and "**11**", which are

$$g^{00} = c^2$$
$$R^{00} = 3c^2(\ddot{a}/a)$$
$$\mathcal{R} = (6/a^2)\,(a\ddot{a} + \dot{a}^2 + kc^2)$$
$$\Rightarrow G^{00} = -3(c/a)^2\,(\dot{a}^2 + kc^2)$$
$$T^{00} = \rho c^2$$
$$\therefore\ -3(c/a)^2\,(\dot{a}^2 + kc^2) = -\,8\pi G\,\rho c^2 - \Lambda c^2$$

$$\boxed{(\dot{a}/a)^2 + k(c/a)^2 = (8\pi G/3)\,\rho + (\Lambda/3)}$$

$$g^{11} = -c^2 a^2 / (1 - kr^2)$$
$$R^{11} = -c^2\,(a\ddot{a} + 2\dot{a}^2 + 2kc^2) / (1 - kr^2)$$
$$\mathcal{R} = (6/a^2)\,(a\ddot{a} + \dot{a}^2 + kc^2)$$
$$\Rightarrow G^{11} = c^2\,(2a\ddot{a} + \dot{a}^2 + kc^2) / (1 - kr^2)$$
$$T^{11} = P a^2 / (1 - kr^2)$$
$$\therefore\ c^2\,(2a\ddot{a} + \dot{a}^2 + kc^2) = -\,8\pi G P a^2 + \Lambda c^2 a^2$$

$$\boxed{(2a\ddot{a} + \dot{a}^2 + kc^2)/a^2 = -\,8\pi G\, P/c^2 + \Lambda}$$

⇨ FRW Model for the Einstein's Field Equations

✓ The first equation is an **energy equation**, and relates a **global dynamics (expansion/contraction)** to a combination of **energy content** and **global curvature** of the Universe

$$(\dot{a}/a)^2 = (8\pi G/3)\rho + \Lambda/3 - k(c/a)^2 \qquad (1)$$

the second one is a **equation of motion**

$$2\ddot{a}/a + (\dot{a}/a)^2 + k(c/a)^2 = -\,8\pi G\,(P/c^2) + \Lambda \qquad (2)$$

sometimes this equation is written in the following way, by replacing the first one on it

$$\ddot{a}/a = -\,4\pi G/3\,(\rho + 3P/c^2) + \Lambda/3 \qquad (3)$$

this equation states that the **acceleration** of the expansion **decreases** with **increasing** energy density (**ρ**) and pressure (**P**)

✓ In these equations, ***a*** **(t)**, **ρ(t)** and **P(t)** are independent functions, and **k** is the **curvature sign**

⇨ First Solutions: Einstein Solution - Static Universe

✓ The Einstein's Field Equations were formulated in 1915 [1916, Ann. der Phys. 49, 769]
Since they appeared very complicated, Einstein was not sure if the solution would ever be found. He was therefore quite surprised when, only a year later, Karl Schwarzschild discovered a solution in the case of a static spherically symmetric metric
(the famous Schwarzschild solution for "simple" **black-holes**)

✓ Einstein himself found, in 1917 [Publ. Acad. Wiss. 142], the first **"cosmological" solution** (that is, for the whole Universe), considering:

- the **Cosmological Principle**: $\rho(x_i,t) = \rho(t)$
- **no pressure** (on mean): $P = 0$
- an **static** Universe (no contraction/expansion): $a(t) = cte = a_0$, $\rho(t) = cte = \rho_0$
- **finite** and **unbound** topology: $k = +1$

In this case, $\dot{a} = \ddot{a} = 0$

Eq. (3):

$$\ddot{a}/a = -\,4\pi G/3\ (\rho + 3P/c^2)$$

$$0 = -\,(4\pi G/3)\ \rho_0\ !$$

✓ This was the reason why Einstein introduced his **cosmological constant**, in order to "correct" his law of gravitation

Eq. (3):

$$\ddot{a}/a = -\,4\pi G/3\ (\rho + 3P/c^2) + \Lambda/3$$

$$\Lambda = 4\pi G\ \rho_0$$

⇨ First Solutions: De Sitter Solution - Empty Universe

✓ In the same year that Einstein proposed his solution for a static Universe, W. de Sitter discussed a solution for an **empty Universe**, that is:

- **no matter** or **pressure**: $\boldsymbol{\rho = P = 0}$
- **flat** topology: $\mathbf{k = 0}$

Eq. (1):

$$(\dot{a}/a)^2 = (8\pi G/3)\rho + \Lambda/3 - k(c/a)^2$$
$$(\dot{a}/a)^2 = (\Lambda/3)$$
$$H(t) \equiv \dot{a}/a = (\Lambda/3)^{1/2}$$

✓ This is also the case for $\boldsymbol{\rho = \text{constant} = \rho_0}$

$$(\dot{a}/a)^2 = (8\pi G/3)\,\rho_0 + (\Lambda/3)$$
$$H(t) = \sqrt{(8\pi G/3)\,\rho_0 + (\Lambda/3)}$$

✓ The solution for these equations is

$$H(t) = \dot{a}/a = da\,/\,(a\,dt)$$
$$H(t)\,dt = da\,/\,a = d\ln a$$
$$\boxed{a \propto e^{Ht}}$$

✓ Eddington characterized the de *Sitter Universe* as "**motion without matter**", in contrast to the static *Einstein Universe* which was "**matter without motion**"

✓ Universes with **exponential expansion** are nowadays called ***inflationary***

⇨ Steady-State Solution: Continuous Matter Creation Universe

- ✓ A model that was popular in 60's was the **steady-state** one, proposed in 1948 by Bondi & Gold [MNRAS 108, 252], based on the statement of the **Perfect Cosmological Principle** (*the Universe is isotropic in space and homogeneous in space and **time***)

- ✓ this theory drew its motivation from the philosophical problems of *big-bang* models (which begin in a singularity at t = 0, and for which earlier times have no meaning)

- ✓ in this model the **Hubble constant** is really constant, the **expansion** is **exponential**, like in the de Sitter model, and necessarily k = 0

- ✓ however, it contains matter and, its mean density **(ρ) must be constant**, even though the Universe is **expanding**. So, it violates energy conservation: it **requires continuous creation of matter!** F. Hoyle has considered a modified General Relativity that no longer conserves mass, and found a way to obtain the steady-state Universe with $\boldsymbol{\rho = \rho_{crit}}$

- ✓ the CMBR observation **ruled out** this possibility

⇨ Friedmann-Lemaître Solutions

- ✓ The **complete solution** for the application of Einstein's Field Equations to the whole Universe was derived by A. Friedmann in 1922, and confirmed by an independent formulation in 1927 by G. Lemaître [G. Ann. Soc. Sci. Bruxelles A47, 49] That is the solution presented in the beginning of this lecture

- ✓ It uses the mathematical framework proposed independently by H.P. Robertson (in 1929) and A.G. Walker (in 1936) – the **Robertson-Walker metric**

- ✓ The Friedmann-Lemaître Universe is **dynamic**, with the a **(t)** dictating its expansion/contraction. Note that this was proposed before the discovery of the **Universe expansion** by E. Hubble [1929, Proc. Nat. Acad. Sci.15]

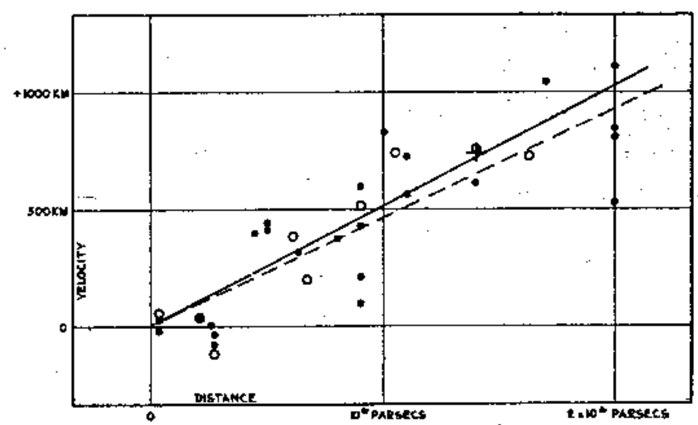
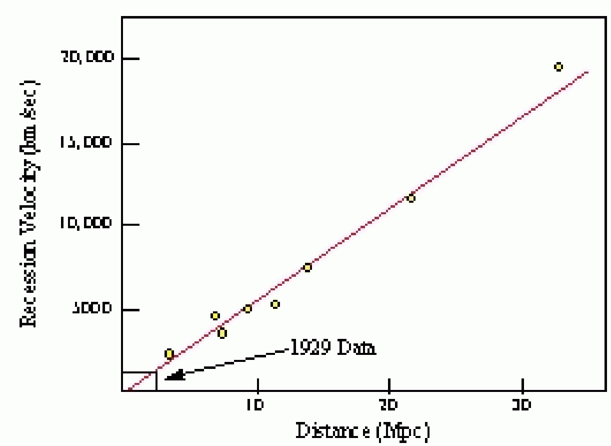
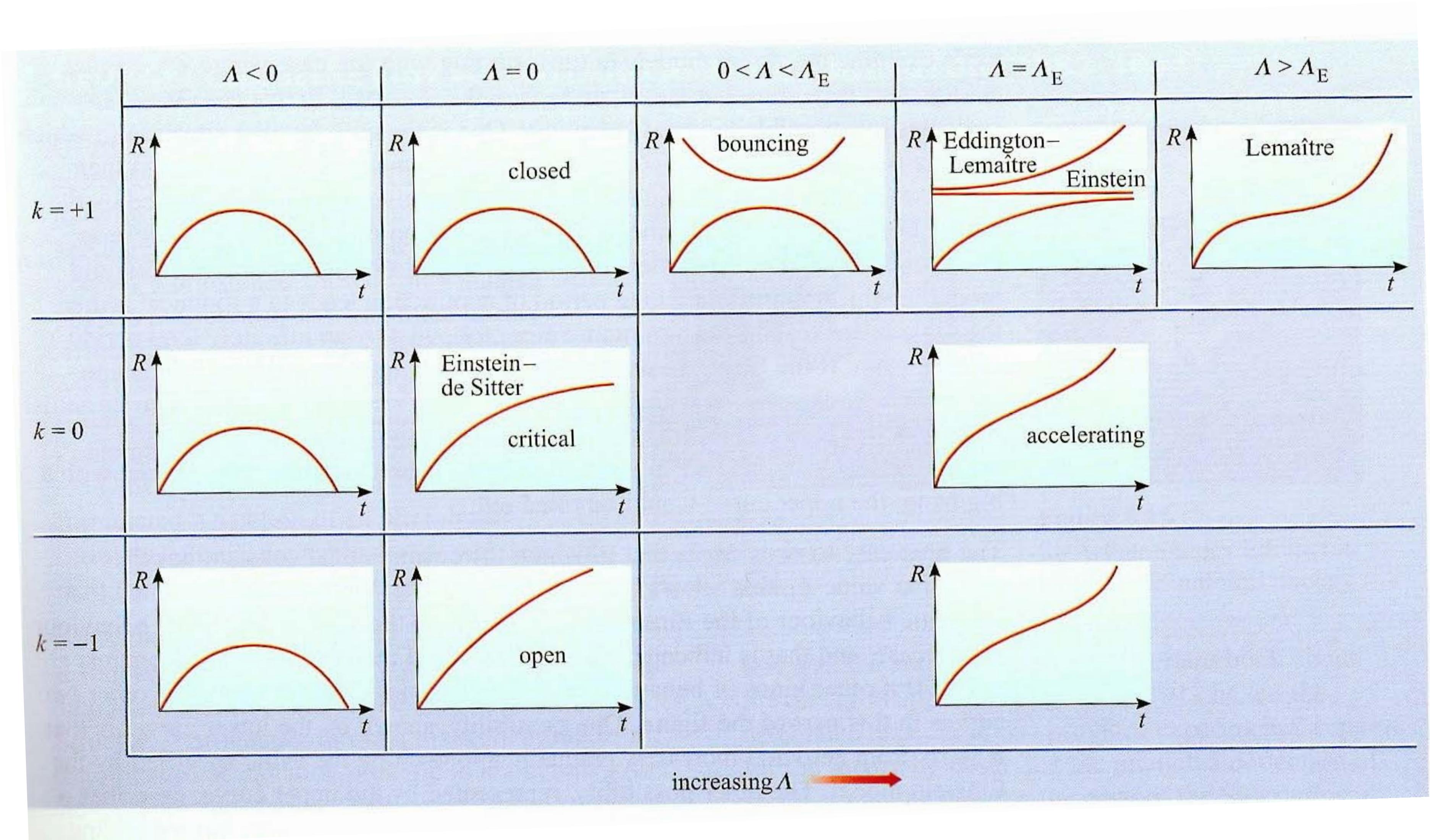


FIGURE 1

Hubble & Humason (1931)



⇨ Friedmann-Lemaître Solutions



⇨ First Law of Thermodynamics

✓ If one derives the Friedmann energy equation and then replaces the equation of motion

Eq. (1): $a'^2 + kc^2 = (8\pi G/3)\,\rho a^2 + (\Lambda a^2/3)$

$$2a'a'' = (8\pi G/3)\,(\rho' a^2 + 2\rho a\, a') + 2(\Lambda/3) a\, a' \qquad (:\, 2aa')$$

$$a''/a = (4\pi G/3)\,(\rho' a/a' + 2\rho) + (\Lambda/3)$$

Eq. (3): $-(4\pi G/3)\,(\rho + 3P/c^2) = (4\pi G/3)\,(\rho' a/a' + 2\rho)$

$$-3\rho - 3P/c^2 = \rho'(a'/a)$$

$$\boxed{\rho' = -3\,(a'/a)\,(\rho + P/c^2)}$$

this can be easily interpreted as the **adiabatic** first law of thermodynamics,
$dQ = dU - P\,dV = d(\rho a^3) - P d(a^3) = 0$

✓ Applying this equation to the (**non-relativistic**) **matter** content of the Universe, that is, considering **$P = 0$** (the substratum fluid is called "**dust**"); to the (relativistic) **radiation** (**$P = \rho c^2/3$**); and to the "**vacuum energy**" (cosmological constant, **$P = -\rho c^2$**), we find

$\rho_m' = -3\,(a'/a)\,\rho_m$
$d\rho_m/\rho_m = -3\, da/a$

$\rho_m \propto a^{-3}$

$\rho_{rad}' = -3\,(a'/a)\,(\rho_{rad} + \rho_{rad}/3)$
$d\rho_{rad}/\rho_{rad} = -da/a\,(4)$

$\rho_{rad} \propto a^{-4}$

$\rho_\Lambda' = -3\,(a'/a)\,(\rho_\Lambda - \rho_\Lambda)$
$d\rho_\Lambda = 0$

$\rho_\Lambda \propto$ constant

⇨ Equation of State

✓ To complete the Friedmann's equations one needs **another independent relation** to solve a(t). This is usually given by the ***Equation of State***, of the form **P = P(ρ)**.
In Cosmology, the following simple relation is assumed

$$P = \omega \rho c^2$$

substituting it on the conservation equation

$$\rho' = -3 \, (a'/a) \, (\rho + \omega \rho)$$
$$d\rho/\rho = -3 \, da/a \, (1 + \omega)$$

$$\boxed{\rho \propto a^{-3(1+\omega)}}$$

thus, the **dust** (baryonic + collisioness dark matter), the **radiation** and the **vacuum energy** correspond to **ω = 0**, ⅓ and **-1**, respectively