Lecture 17
Structure Formation I – Power Spectra

- Power Spectra of Fluctuations
  & Origin of Inhomogeineities
    - Primordial Power Spectrum
    - Transfer and Growth Functions
    - Anisotropies in the CMBR
    - Current Power Spectrum
- Linear Evolution of Perturbations
- Non-Linear Evolution of Perturbations
- Simulations of Structure Formation

⇨ Primordial Power Spectrum

- Since the early evolution of inhomogeneities is a **linear process**, we can make a **Fourrier** decomposition of the **fluctuations**

$$\delta(\mathbf{x},t) = 1/(2\pi)^3 \int d^3\mathbf{k}\ \ \delta_k(t)\ \ e^{i\mathbf{k}.\mathbf{r}}$$

- The **Power Spectrum** of the fluctuations (i.e., the distribution of amplitudes of the scale modes) is defined by the powers $|\boldsymbol{\delta}_{\mathbf{k}}\mathbf{(t)}|^2$:

$$P(k) \equiv <|\delta_k(t)|^2>$$

- With the lack of more accurate knowledge of the PS, one assumes, for simplicity, that it is specified by a **power law** (that is, it contains any preferred length scale, otherwise we should then be compiled to explain this feature):

$$P(k) \propto k^n$$

where **n** is the **spectral index**.

- By definition, P(k) has dimensions of $(\text{length})^3$. However, it is usual to express the PS as a **dimensionless** function by multiplying it by $\mathbf{k^3}$, defining the **rms variance**

$$\Delta^2(k) = [k^3/\ 2\pi^2]\ P(k) \propto k^{n+3}$$

⇨ Primordial Power Spectrum: Harrison-Zel'dovich Spectrum

✓ The **spectral index** may be constrained to a certain range of values: if **n** < **–3**, the Univ would be very inhomogeneous on the **largest scales today**; if **n** > **4**, the Univ would be very **inhomogeneous at early times** (which disagrees with the primordial helium abundance measurements); **n = 0** corresponds to **white noise** (Poisson PS).

✓ E. R. Harrison [1970, Ph. Rev. D 1, 2726] and Ya. B. Zel'dovich [1972, MNRAS 160, 1] proposed that there should be no particular scale to $\boldsymbol{\delta_H}$ (a density perturbation at the horizon crossing), since the early Univ was **acausal on all scales** before horizon crossing – in other words, that the Univ must always looks the same (be **self-similar**) when viewed on the scale of the horizon. This means that the PS should be **scale-invariant** (equal amplitude on all scales), with **n = 1**

$$\Delta_H = \Delta(d_H) \propto a^2\, k^{(n+3)/2} \propto d_H^{\,2}\, d_H^{\,-(n+3)/2} \propto d_H^{\,(1-n)/2} \propto 1 = \kappa$$

In addition, the fluctuations **amplitude** (κ) cannot be too large, or else the CMBR would be **too anisotropic**; and cannot be too small, or **galaxies would not have formed** by the present time. These considerations require $\boldsymbol{\kappa \sim 10^{-5}}$

✓ Today we refer to models in which **n ≠ 1** as models having ***tilt*** or **tilted spectra**

⇨ Primordial Power Spectrum: Topological Defects

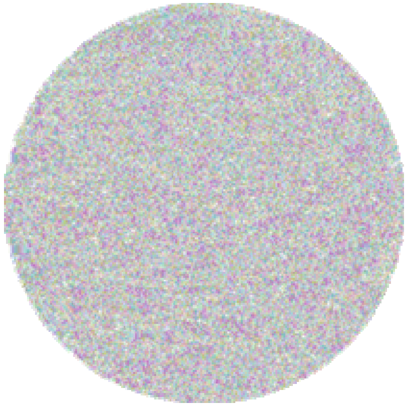
✓ There are **2 theories** that, in principle, can **predict the initial fluctuations** (and their power spectrum) necessary for structure formation: *topological defects* and *inflation*

✓ **Topological defects** appear when there is a symmetry breaking phase transition (that is, when a scalar field has non zero value). They can be of 4 types, depending on the number of components of the scalar field:

- one component (real) → produces 2D defects, called "***domain walls***"
- two components (one real and the other imaginary) → 1D defects, called "***cosmic strings***"
- three components (isovector) → point defects, called "***monopoles***"
- more than 3 components → unstable defects, called "***textures***"

this theory predicts that the stable defects are the **seeds** for the fluctuations.
The most promising of these defects have been the **cosmic strings**, but they have the disadvantage that they produce **non-Gaussian fluctuations**, which is in discordance with the CMBR results.

⇨ Primordial Power Spectrum: Inflation



- ✓ **Inflation** predicts that **quantum mechanical fluctuations**, in the scalar field that drives the exponential expansion, are the **seeds** for the matter-energy perturbations. The produced fluctuations are **Gaussian** (randomly distributed), with zero mean.
  The respective PS has the following shape

$$P_{infl}(k) = (50\ \pi^2)/(9\ k^3)\ (k\ /\ H_0)^{n-1}\ \delta_H^{\ 2}\ (\Omega_M\ /\ D_0)^2$$

where $\boldsymbol{\delta_H}$ are the density perturbations **amplitude** at horizon cross, and **D** is the **growth function** of perturbations

- ✓ The inflationary models also predict that the PS of matter-energy fluctuations is almost **scale-invariant** (or **scale-free**) as the fluctuations cross the Hubble radius (Harrison-Zel'dovich spectrum)

⇨ Transfer and Growth Functions

- ✓ Although the initial PS (distribution of fluctuations as they enter the horizon) is a pure power law, perturbation growth results in a modified PS.
- ✓ While on **large scales** the power spectrum follows a simple **linear evolution**, on small scales it changes shape due to the **additional non-linear gravitational growth** or perturbations and it results in a bended spectrum.
- ✓ The evolution on the **linear regime** comprehends the **early evolution** (between the horizon cross and the decoupling time), which may be incorporated in the PS as a **transfer function**, and **later evolution** (before the scales enter the nonlinear regime, which depends on the scale), which is called **growth function**

$$P(k,a) = P_H(k)\ (9/10)\ T(k)\ [D(a)/a\ ]$$

- ✓ Since **T(k)** depends on the mixture of matter (both collisionless **dark particles** and **baryonic plasma**) and energy (relativistic **collisionless** ν and **collisional** γ) components of the Universe, which does not behave as a simple fluid, the calculation of results for this function is a technical challenge. Accurate results require a solution of the **Boltzmann equation** for each species to follow the evolution in detail [see, for example, Dodelson 2003, chaps. 4-7].

- ✓ **Analytical approximations** have been proposed. Since **DM** is the most important component concerning to structure formation, let us see the fitting forms to T(k) for the two main possible kinds of DM: **Cold** and **Hot**

⇨ Transfer Function

✓ Bond & Efstathiou [1984, ApJL 285, L45] and Bardeen et al. [1986, ApJ 304, 15, BBKS] proposed, for **CDM**

$$T(k) = \frac{\ln(1+2.34q)}{2.34q} \frac{1}{[1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4]^{1/4}} .$$

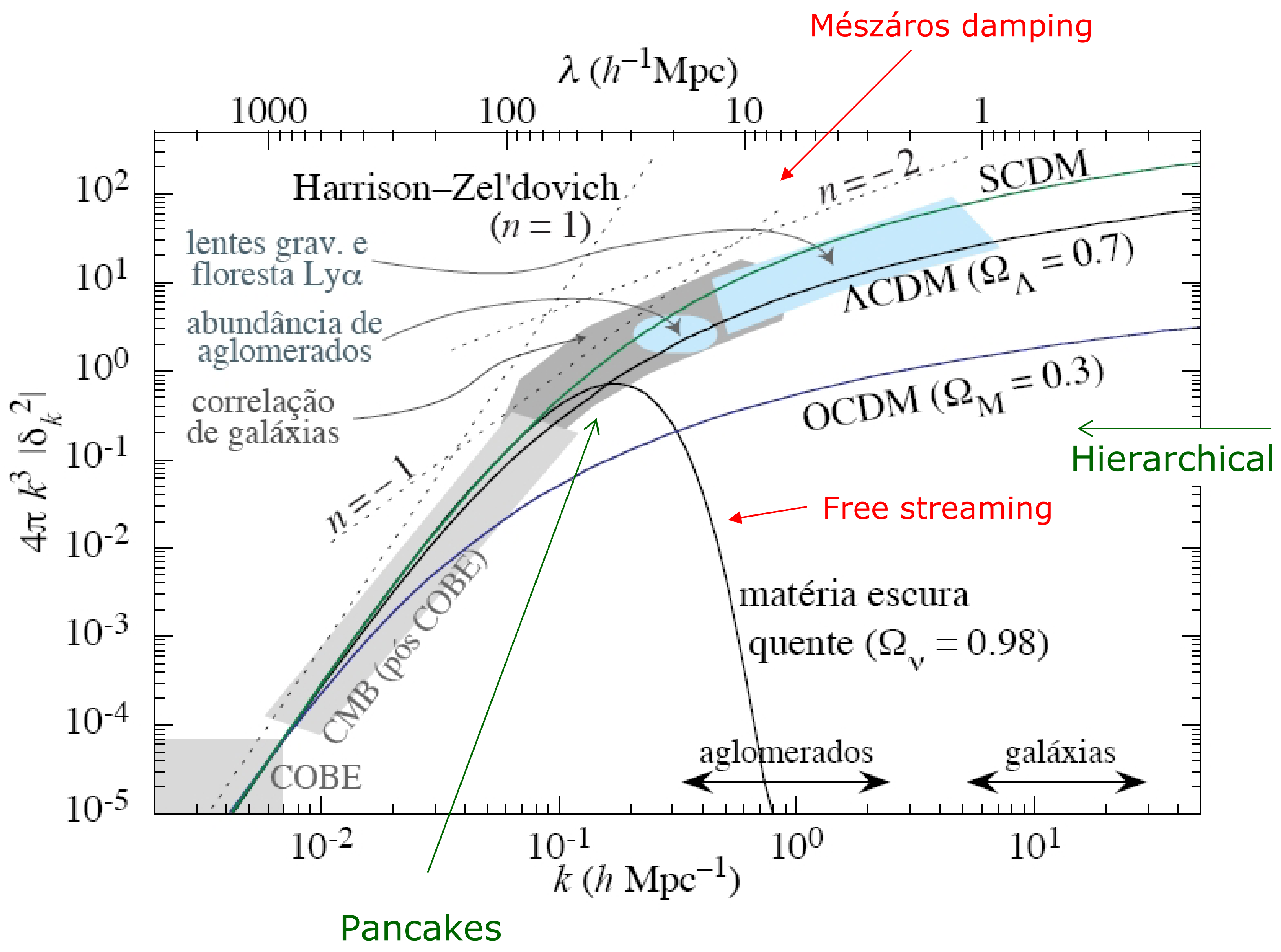
where $\mathbf{q = k/(\Gamma\, \Omega_M\, h^2)}$ and $\mathbf{\Gamma = (T_{CMB}/2.73K)^{-2} \exp\{-\Omega_B[1+\sqrt{(2h)}/\Omega_M]\}}$

✓ For **HDM**, the fitting function [Bond & Szalay 1983, ApJ 276, 443] is

$$T(k) = \exp[-4.61\ (k/k_{FS})^{3/2}]$$

where $\mathbf{k_{FS} = 0.16\ (m_\nu / 30eV)\ Mpc^{-1}}$

⇨ Transfer Function



⇨ Transfer Function

✓ The effect of the other components on the **T(k) of CDM** can be summarized as follows:

- BARYONS – they slightly **suppress** T(k) on **small scales**
  – they produce small **oscillations** around k ~ 0.1 *h* Mpc^{-1}

- MASSIVE ν – they also **suppress** T(k) on **small scales** (if they have much mass, they behave as **HDM**, and so their fast motion damp small perturbations)

- DARK ENERGY – moves the epoch of equilibrium to later times, **changing** the **turn over** scale (the one that enters the horizon at equality) of the PS (to larger scales)
  – changes the **normalization** of the PS (the amplitude increases as the matter decreases, and more Λ exists for a flat model)
  – the **growth factor** changes, since it depends on H(*a*), which depends on Λ

⇨ PS from the CMBR

- ✓ We can divide the effects that compose the CMBR in
  - large fluctuations (> 10°) – consistent with Harrison-Zel'dovich spectrum for these scales
  - intermediate fluctuations (~ 1°) – dominated by effects of **potential fluctuations** at the surface of last scattering (Sachs-Wolfe Effect)
  - small fluctuations (< 1°) – just below a degree, the **Doppler effect** begins to dominate. The plasma at recombination is moving, so that there is a net shift in photon energy which depends on the peculiar velocity. Soon the CMBR fluctuations are dominated by **adiabatic effects** (hotter regions recombine later and are less redshifted today)
- ✓ The resulting fluctuation spectrum on the sky is, consequently, a function of the cosmological parameters.

[see, for example, Padmanabhan 1993, chaps. 6; or Dodelson 2003, chap. 8]

⇨ PS from the CMBR

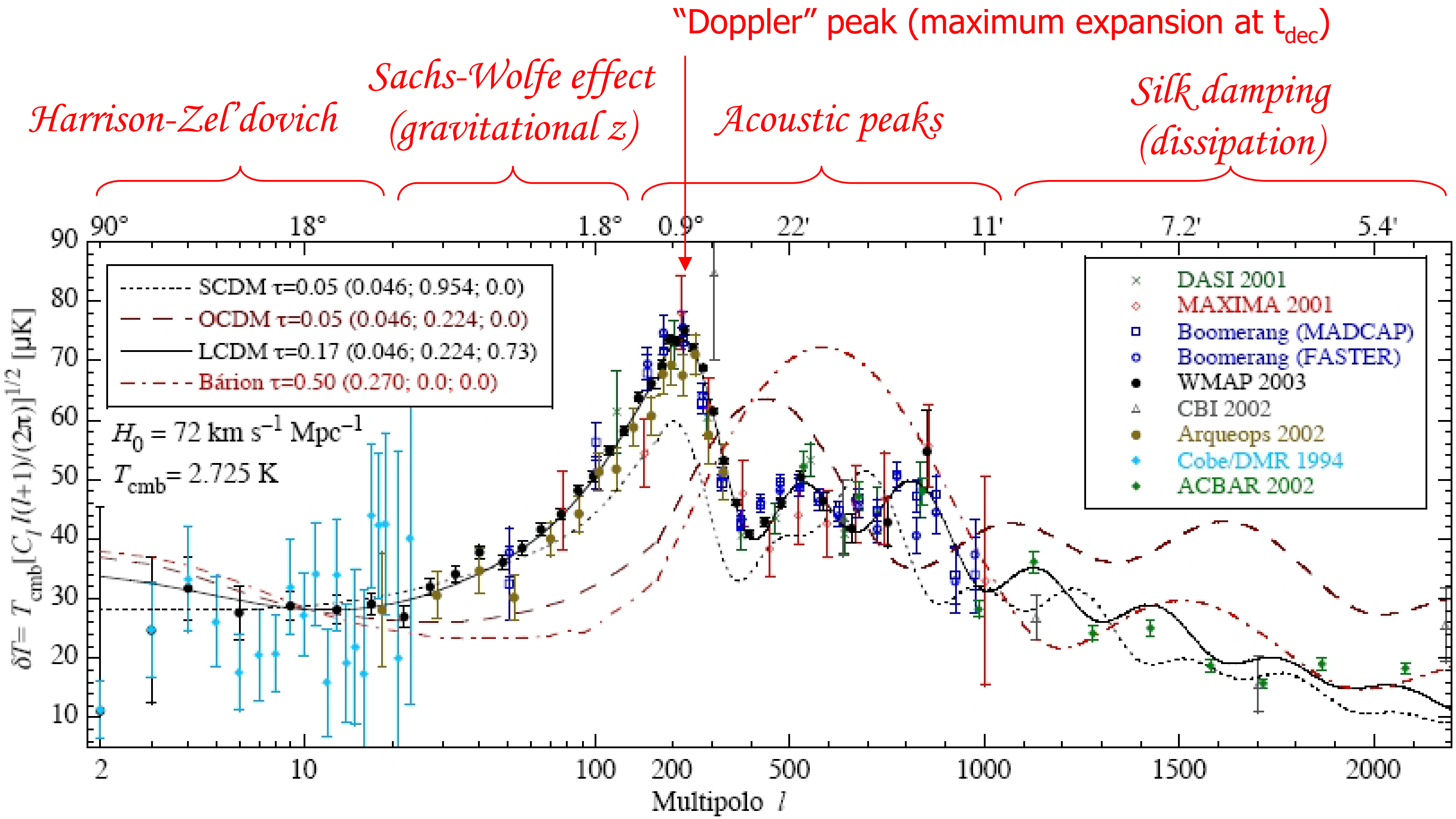


Figura 38: Espectro de potência multipolar (angular) da CMB. Os parâmetros dos modelos teóricos (alguns exemplos, calculados com `cmbfast`) estão na legenda à esquerda. Os números entre parênteses são, respectivamente: Ω_B, Ω_{cdm} (apenas matéria escura) e Ω_Λ. A densidade total de matéria é $\Omega_M = \Omega_B + \Omega_{cdm}$.

⇨ Growth Function

✓ For finding the growth function, one needs to integrate the **growth equation** (next class).

$$D(a) = (5/2)\,\Omega_M\, H(a)/H_0 \int_{0\to a} [(a/H_0)\, H(a)]^{-3}\, da$$

✓ No matter what constitutes the matter-energy budget today, all modes have experienced the **same growth factor** (after decoupling). This uniform growth is a direct result of the fact that **CDM** has **zero pressure**.

✓ If the Univ is **flat** and **matter dominated**, the **growth factor** is simply **equal** to the **scale factor**. In both **open** and **dark energy** cosmologies, though, **growth is suppressed** at late times: this means that **structure** in a open or dark energy Univ **developed much earlier** than in a EdS Univ and so, there has been relatively **little evolution at recent times**.

⇨ Current PS measurements

✓ The bending of PS due to non-linear growth of perturbations results in **P(k)** $\propto$ $\mathbf{k^{n-4}}$ on smaller scales

✓ There are several ways to represent the spectrum of perturbations beyond its Fourrier decomposition

- fractional density excess **δ**, as a function of scale

$$\rho(\mathbf{x},t) = \langle\rho\rangle\,[1 + \delta(\mathbf{x},t)]$$

- rms **mass fluctuations** inside a randomly placed **sphere** of radius R, $\Delta(\mathcal{M})$

$$\sigma(\mathcal{M}) = \langle(\delta\mathcal{M}/\mathcal{M})^2\rangle = V/(2\pi)^3 \int d^3k\; W^2(kR)\,|\delta_k|^2$$

$$W(kR) = (3/4\pi R^3) \int_{sphere} d^3\mathbf{x}\; e^{i\mathbf{k}.\mathbf{x}} = 3/(kR)^3\,[\sin(kR) - kR\cos(kR)]$$

$$\Delta^2(k) \propto k^{n+3}\;; \qquad k = 2\pi/\lambda \propto \mathcal{M}^{-1/3}\;;$$

$$\Delta^2(\mathcal{M}) \propto (\mathcal{M}^{-1/3})^{n+3}\;; \qquad \therefore\; \Delta(\mathcal{M}) \propto \mathcal{M}^{-(n+3)/6}$$

- **velocity** induced by fluctuations

$$\delta v^2 = V/(2\pi)^3 \int d^3k\; |v_k|^2\, e^{i\mathbf{k}.\mathbf{r}}$$

$$\delta v \propto \mathcal{M}^{-(n+1)/6}$$

⇨ Current PS measurements: Correlation Function

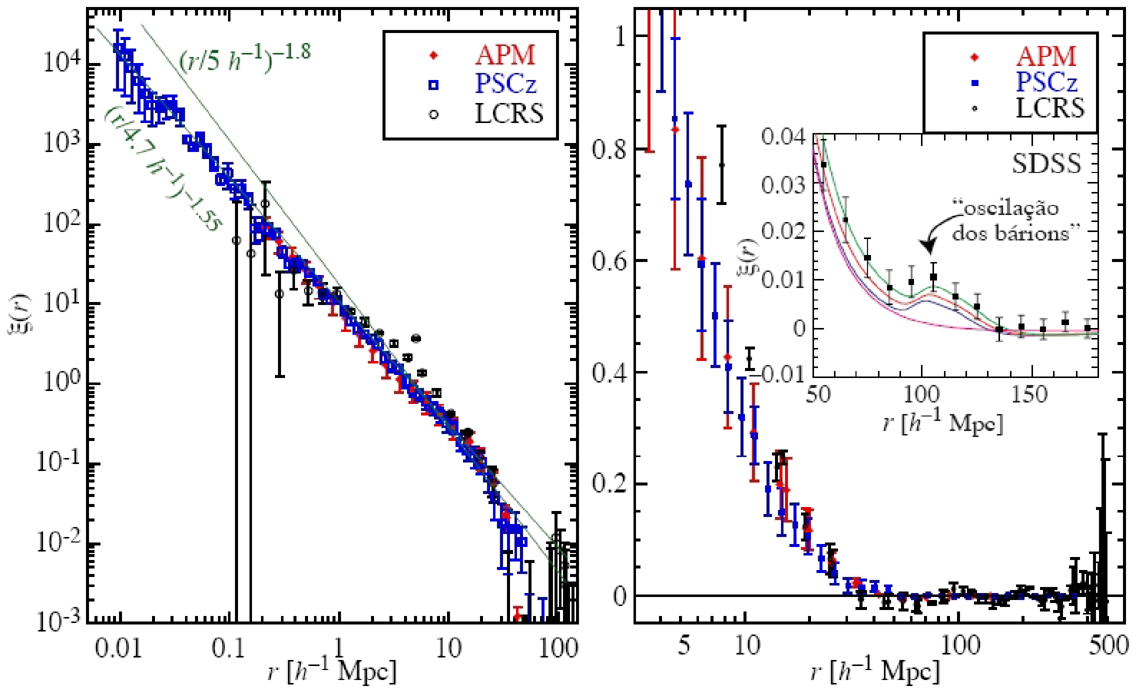
- **autocorrelation function** (that is the Fourrier transform of the PS)

$$\xi(r) = < \delta(\mathbf{x})\ \delta(\mathbf{x} + \mathbf{r}) >$$

$$\xi(r) = V/(2\pi)^3 \int d^3k\ |\delta_k|^2\ e^{i\mathbf{k}.\mathbf{r}}$$

- galaxy **autocovariance function**

$$d\mathcal{P} = n_{gal}\ dV$$
$$d\mathcal{P}_{12} = n_{gal}^2\ dV_1\ dV_2$$
$$d\mathcal{P}_{12} = n_{gal}^2\ [1 + \xi(r)]\ dV_1\ dV_2$$



⇨ Current PS measurements

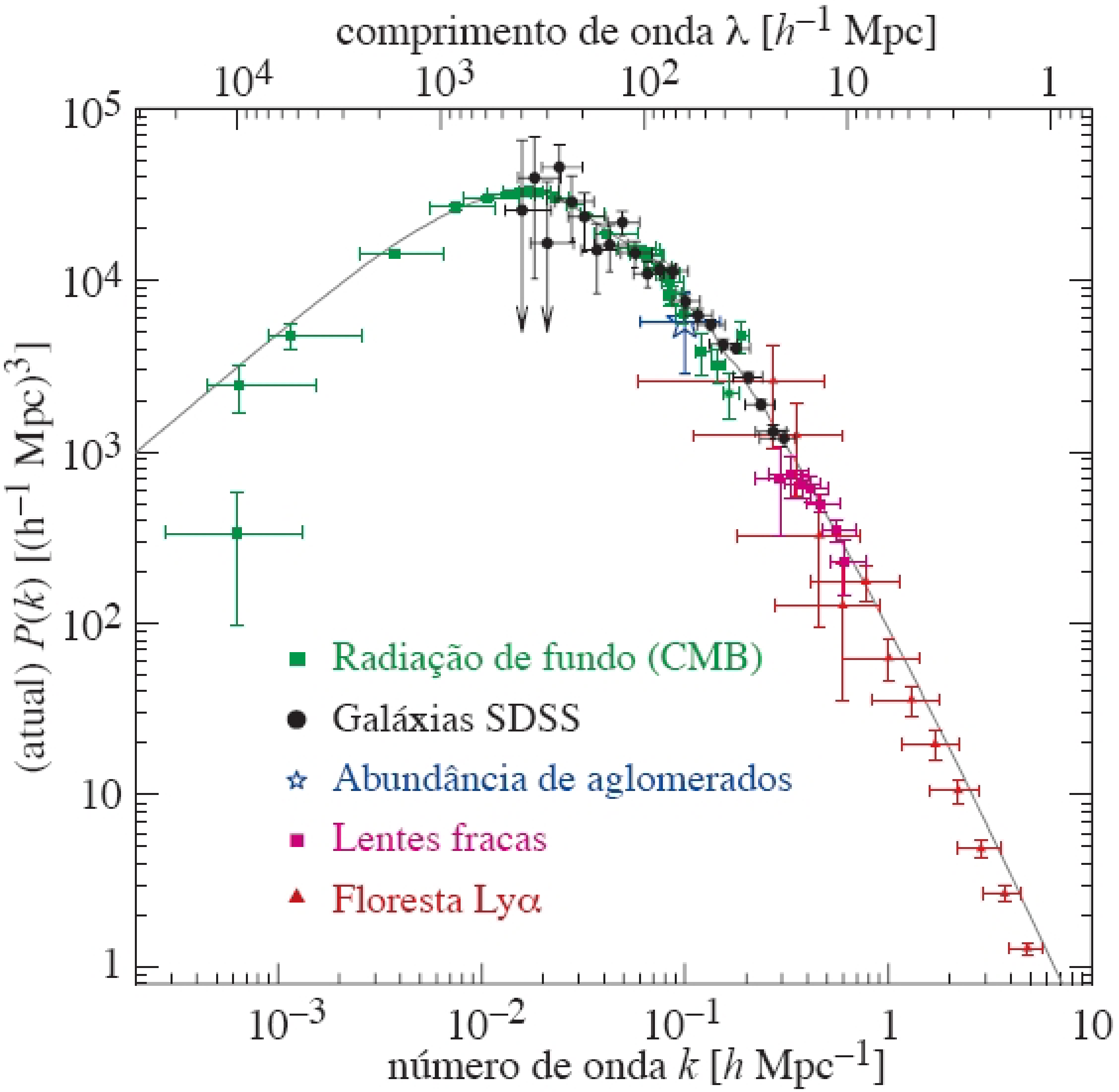


Figura 50: Espectro de potência das flutuações de densidade determinados a partir de diversas técnicas (note que cada uma é mais sensível para um certo intervalo de número de onda k). O traço contínuo representa um modelo com $\Omega_\Lambda = 0{,}72$, $\Omega_M = 0{,}28$ e $h = 0{,}72$, com perturbações iniciais independente de escala ($n = 1$) e profundidade óptica até a superfície de último espalhamento $\tau = 0{,}17$ (cf. seção 11). Figura tirada de Tegmark et al. (2004).

⇨ References

Books:

- T. Padmanabhan 1993, *Structure Formation in the Universe*, Cambridge Univ. Press
- M. Longair 1998, *Galaxy Formation*, A&A Library – Springer
- S. Dodelson 2003, *Modern Cosmology*, Academic Press
- B. Ciardi & A. Ferrara 2004, astro-ph 0409018 👍