# Lecture 18
# Structure Formation II – Linear Evolution

- Power Spectra of Fluctuations
  & Origin of Inhomogeinities
- Linear Evolution of Perturbations
  - Derivation of the Growth Equation
  - Jeans Scale
  - Horizon Scale
  - Damping
- Non-Linear Evolution of Perturbations
- Simulations of Structure Formation

# ⇨ Linear Evolution of Perturbations

- ✓ We assume that, at some time in the past, there were **small deviations from homogeneity** in our Universe. These deviations can grow due to **gravitational instability** over a period of time and, eventually, form **galaxies**, **groups**, **clusters**, **superclusters**, ...

- ✓ As long as these inhomogeneities are **small**, their growth can be studied by the **linear perturbation theory**. Once the deviations from the smooth Universe become large, linear theory fail and we have to use other techniques to understand their **nonlinear evolution**.

- ✓ The analysis of the **growth of perturbations** in the Universe is very similar to the classic **J. Jeans** (1902) calculation of a **spherical gravitational collapse**, except that it occurs within an **expanding** Universe. The first of such calculation for an expanding Universe was done by **Lifschitz** (1946), and a short time later, **Bonner** showed how the calculation can be done using Newtonian Cosmology (for perturbations **smaller than the horizon** and in a **matter dominated** regime)

- ✓ The matter-energy content of the smooth Universe has 3 main components: ***relativistic*** "matter" [**photons** ($\gamma$) and **neutrinos** ($\nu$)], "dusty" **baryons** (**B**) and "collisionless" **dark matter** (**DM**). In the radiation domination era the 3 components are coupled (the $\nu$ decouple before, but if they are about massless their role is minor). Then, about $\mathbf{t_{eq}}$ the DM decouple and at $\mathbf{t_{dec}}$ B and $\gamma$ decouple. Thus, the evolution of each of these components is quite different and the growth of inhomogeneities in each of them should be **treated separately**.

## ⇨ Derivation of the Growth Equation

✓ The distribution of matter in the Universe can, to some approximation, be described by the hydrodynamics of a non-viscous, non-static **fluid**. An ordinary fluid is dominated by the material pressure, but in the fluid of the Universe, 3 effects are competing:

- **gravity attraction**
- **radiation pressure**
- **density dilution due to the Hubble flow**

✓ We will derive the Newtonian **Growth Equation**, from the basic fluid equations in an expanding Universe. We will use *Lagrangian* coordinates (the motion of a particular fluid element is followed) instead of *Eulerian* coordinates (the partial derivatives describe the variation of the quantities at a fixed point in space), since for the cosmological problem we have a **comoving fluid element**.

| *Eulerian* coordinates | *Lagrangian* coordinates |
|---|---|
| $\partial\rho/\partial t + \nabla.(\rho\mathbf{v}) = 0$ | $d\rho/dt + \rho\,\nabla.\,\mathbf{v} = 0$ |
| $\partial\mathbf{v}/\partial t + (\mathbf{v}\,.\nabla)v = -(1/\rho)\,\nabla P - \nabla\Phi$ | $d\mathbf{v}/dt = -(1/\rho)\,\nabla P - \nabla\Phi$ |
| $\nabla^2\Phi = 4\pi G\,\rho$ | $\nabla^2\Phi = 4\pi G\,\rho$ |

where $d/dt = \partial/\partial t + (\mathbf{v}\,.\nabla)$

# ⇨ Derivation of the Growth Equation

✓ Commoving coordinates

$$\mathbf{x} = \mathbf{r}\,/a \quad \Rightarrow \quad \mathbf{r} = a\,\mathbf{x}$$
$$\delta\mathbf{r} = \delta(a\,\mathbf{x}) = \delta a\ \ \mathbf{x} + a\ \ \delta\mathbf{x}$$
$$\delta\mathbf{r}/\delta t = \delta a\,/\delta t\ \ \mathbf{x} + a\ \ \delta\mathbf{x}/\delta t$$
$$\mathbf{u} = \mathring{a}\,(\mathbf{r}/a) + \mathbf{v}_{pec}$$
$$\mathbf{u} = H\ \mathbf{r} + \mathbf{v}_{pec} \quad \Leftrightarrow \quad \mathbf{v} = \mathbf{v}_0 + \delta\mathbf{v}$$

✓ Equation of motion

$$\delta^2\mathbf{r}/\delta t^2 = a''\ \mathbf{x} + a'\ \mathbf{x}' + a'\ \mathbf{x}' + a\,\mathbf{x}''$$
$$= a''/a\ \mathbf{r} + 2\ a'\ \mathbf{u} + a\,\mathbf{u}'$$

✓ Newton

$$\delta^2\mathbf{r}/\delta t^2 = -\nabla(\Phi_0 + \Phi)\ /\ a$$

$$a''/a\ \mathbf{r} = -\nabla\Phi_0\ /\ a$$
$$2\ a'\ \mathbf{u} + a\,\mathbf{u}' = -\nabla\Phi\ /\ a$$

$$\Rightarrow -\nabla\Phi\ /\ a^2 = \delta\mathbf{u}/\delta t + 2\ H\ \mathbf{u}$$

# ⇨ Derivation of the Growth Equation

- ✓ Unperturbed solutions (zero order)

$$d\rho_0/dt + \rho_0\, \nabla .\, \mathbf{v}_0 = 0$$

$$d\mathbf{v}_0/dt = -\, (1/\rho_0)\, \nabla P_0 - \nabla\Phi_0$$

$$\nabla^2\Phi_0 = 4\pi G\, \rho_0$$

- ✓ Perturbation

$$\delta = \delta\rho/\rho_0$$

- ✓ 1st order perturbations

$$\rho = \rho_0 + \delta\rho = \rho_0\,(1 + \delta)$$
$$\mathbf{v} = \mathbf{v}_0 + \delta\mathbf{v}$$
$$P = P_0 + \delta P$$
$$\Phi = \Phi_0 + \delta\Phi$$

- ✓ Adiabatic perturbations

$$\partial P/\partial\rho = c_s^{\,2}$$

[see, e.g., Peacock 2003, astro-ph 0309240]

# ⇨ Derivation of the Growth Equation

$$d^2\delta/dt^2 + \underbrace{2\,H\,d\delta/dt}_{\textit{Expansion dilution}} = [\underbrace{4\pi G\,\rho_0}_{\textit{Gravity}} + \underbrace{c_s^{\,2}/(\rho_0 a^2)\,\nabla_r^{\,2}}_{\textit{Pressure}}]\,\delta$$

✓ Since it is a **linear** equation, we can do a **Fourrier decomposition** of $\delta$

$$\delta(\mathbf{x},t) = \int d^3\mathbf{k}\;\; \delta_k(t)\;\, e^{i\mathbf{k.r}}$$

$$\Rightarrow \qquad d^2\delta_k/dt^2 + 2\,H\,d\delta_k/dt = [4\pi G\rho - 4\pi^2 c_s^{\,2}/\lambda^2]\,\delta_k$$

where $\lambda = 2\pi a/|\mathbf{k}|$. This equation is similar to that of a damped harmonic oscillator; so, if the right-hand side (RHS) is **negative** (pressure dominates over gravity) we expect to find only **damped oscillatory solutions**, while if the RHS is **positive** (gravity dominates over pressure) we expect to find **growing unstable modes**.

✓ Therefore, a necessary and sufficient **condition for instability** is

$$G\rho > \pi c_s^{\,2}/\lambda^2$$

$$\lambda > c_s\,\sqrt{(\pi/G\rho)} = \lambda_J$$

$$\mathcal{M} > (4/3)\pi\,(\lambda_J/2)^3\,\rho = \mathcal{M}_J$$

## ⇨ Growth Equation in the Relativistic Case

$$\underbrace{d^2\delta/dt^2 + H\, d\delta/dt\,[2 - 3\,(2\omega - v^2)]}_{\textit{Expansion dilution}} = \underbrace{(3/2)\, H^2\,[1 - 6v^2 - 3\omega^2 + 8\omega]\,\delta}_{\textit{Gravity}} - \underbrace{(kv/a)^2\,\delta}_{\textit{Pressure}}$$

where $\mathbf{v^2 = \delta P/\delta\rho}$, $\boldsymbol{\omega = P/\rho}$ and $\mathbf{H = (8\pi G/3)}\ \rho$

# ⇨ Solutions to Growth Equation

- State equation $P_i = \omega_i \, \rho_i$

$$v_i^2 = \partial P_i / \partial \rho_i = \omega_i$$

  when $\omega_i$ is constant

- The proper wavelength of any mode will be **bigger than the horizon radius** at sufficiently **early epochs**. They enter the Hubble radius at some time $t_{enter}$.
- An overdense region that is larger than horizon radius **cannot be supported** by its internal pressure (or viscosity, etc) because any changes in pressure are propagated at a speed that is lower than *c*. So, it will grow with time (expand at a lower rate than the Universe around it), albeit slowly
- For $\mathbf{t < t_{enter}}$, $\boldsymbol{\lambda > d_H}$, we should use the **GR perturbation theory** equation to evolve $\boldsymbol{\delta}$

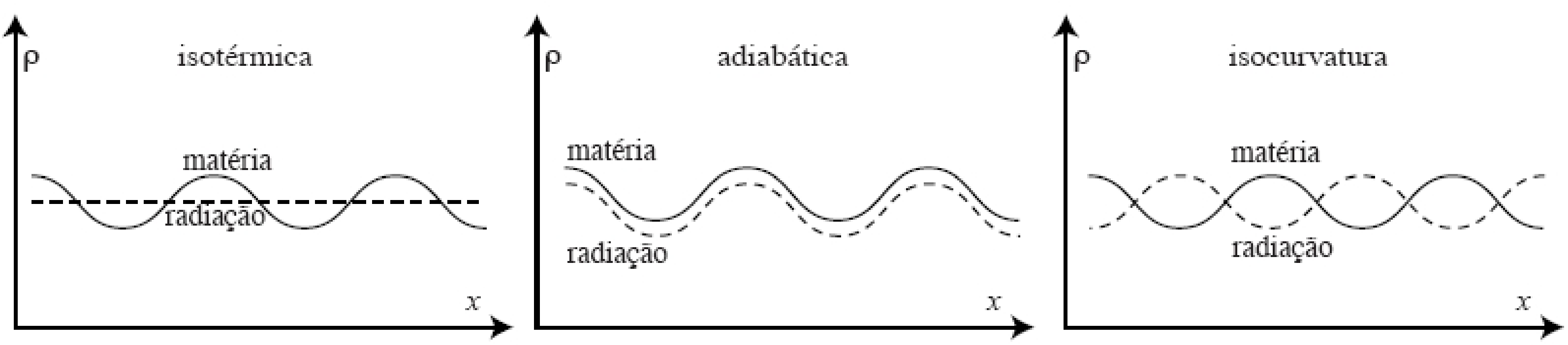
$$\delta \propto \begin{cases} a^2 & , t < t_{eq} \quad \text{(Rad. Dom.)} \\ a & , t > t_{eq} \quad \text{(Mat. Dom.)} \end{cases}$$

- For $\boldsymbol{\lambda < d_H}$, there are two processes that can **prevent the amplitude from growing**: **pressure support** and **expansion dilution**. The second case occurs when the perturbed species is not the dominant species and the **dominant species** (which governs the expansion) is smootly distributed – then the Universe will be expanding too fast for the collapsing region to condense out

- If neither of these processes is operational, then the amplitude will grow.

# ⇨ Type of Perturbations

- ✓ When there is more than one important constituent in the Universe then, in principle, each constituent can be **perturbed separately** and **independently**
- ✓ In a Universe filled mainly with **baryons** and **radiation**, the 2 fundamental modes of perturbation are
  - **isothermical** (or entropic) – during radiation era, photon density is unaltered because of tight EM coupling of $e^-$, $p^+$ and $\gamma$, which hinders them from traveling very far
  - **adiabatic** (or curvature) – matter and radiation are perturbed together in such a way that the specific entropy remains constant.
- ✓ any general perturbation is expressible as a **linear combination** of adiabatic and isothermal fluctuations
- ✓ Another way of describing a perturbation mode is
  - **isocurvature** – the total matter perturbation (including baryons and DM) is of the same amplitude but opposite sign as radiation perturbation; this gives zero perturbation to the curvature of space (they are also adiabatic)



## ⇨ Solutions to Growth Equation

✓ The perturbations in **radiation** neither grow nor decay after they enter the Hubble radius (they **oscillate**). On the other hand, the perturbations in **dark matter** can grow with **ln** $a$ after they enter the horizon and with $a$ **from** $a_{eq}$ **onwards**, while perturbations on **baryons oscillate** when they enter the horizon and grows with $a$ only **after** $a_{dec}$.

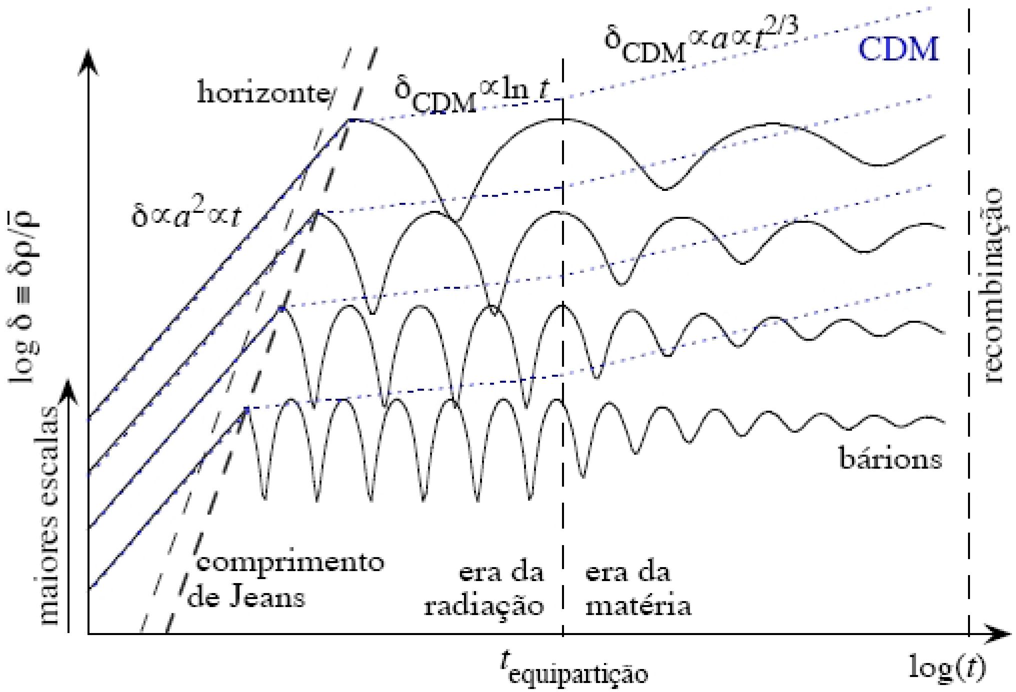



Figura 43: Evolução do contraste de densidade ao entrar no horizonte e se tornar menor que o comprimento de Jeans. A partir deste momento, a perturbação passa a oscilar até o momento da equipartição entre matéria e energia (ver seção 2.4.1). A partir daí, as perturbações de matéria não bariônica fria crescem enquanto que a oscilação das perturbações dos bárions é amortecida (*Silk damping*). Note que as pequenas escalas entram primeiro no horizonte e oscilam com maior freqüência. As curvas estão deslocadas verticalmente para maior clareza.

✓ When the baryons decouple, their perturbation will **feel the perturbed** gravitational potential of DM and will be **driven by it** (we may say that the baryons “fall into” the potential well created by DM. This implies that $\delta_B$ will **grow rapidly** for a short time after $a_{dec}$ and will **equalize** with $\delta_{DM}$, after that both will grow as $a$.

# ⇨ Evolution of Jeans Mass

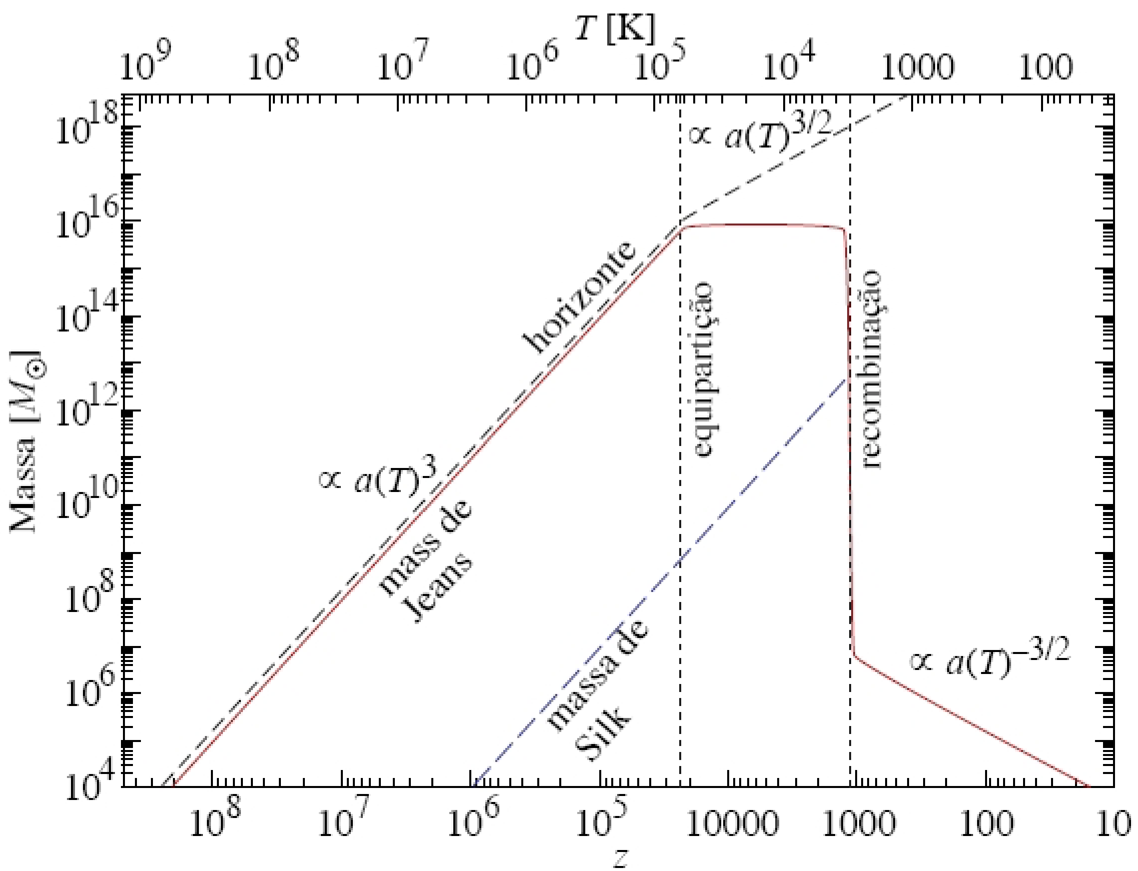



Figura 45: Massa de Jeans em função do *redshift* (e temperatura) para matéria bariônica. Também estão representadas a massa no interior do horizonte ($r_H \approx cH_0^{-1}$) e o limite de massa de Silk, que sobrevive ao "Silk damping"

| | | $\delta_{DM}$ | $\delta_B$ | $\delta_\gamma$ |
|---|---|---|---|---|
| $\lambda > d_H$ | $t < t_{enter}$ | grows $a^2$ | grows $a^2$ | grows $a^2$ |
| $\lambda < d_H$ | $t_{enter} < t < t_{eq}$ | grows ln $a$ | oscillate | oscillate |
| | $t_{eq} < t < t_{dec}$ | grows $a$ | oscillate | oscillate |
| | $t > t_{dec}$ | grows $a$ | grows $a$ | oscillate |

# ⇨ Damping

- ✓ We have considered the matter content of the Universe to be an **ideal fluid**, but this approximation breaks down for **λ smaller** than a particular **critical value**. Below this value **energy is drained away** by certain dissipative processes. The physical origin of dissipation is different for baryons and DM. This happens before recombination.

- ✓ For **collisionless DM**, dissipation occurs through a process called "**free-streaming**": for perturbations smaller than the free-streaming mass, the particles within them are not gravitationally bound and freely stream away.

- ✓ For **baryons**, dissipation arises due to **coupling between radiation and matter**: perturbations in photons and baryons will diffuse away when the photons can escape.

# References

Books:


- T. Padmanabhan 1993, *Structure Formation in the Universe*, Cambridge Univ. Press
- M. Longair 1998, *Galaxy Formation*, A&A Library – Springer