Lecture 19
Structure Formation III – Nonlinear Evolution

- Power Spectra of Fluctuations
  & Origin of Inhomogeineities
- Linear Evolution of Perturbations
- Non-Linear Evolution of Perturbations
  - Spherical Collapse Model
  - Lagrangian approach (Zel'dovich Approximation)
  - Angular momentum and Formation of Discs
  - Formation of Spheroids
- Simulations of Structure Formation

⇨ Non-linear Evolution of Perturbations

- ✓ The **linear perturbation** theory **fails** when the density contrast becomes **nearly unity**. Since most of the observed structures in the Univ (galaxies, clusters, etc) have density contrasts far in excess of unity, their formation can be understood only by a fully **nonlinear theory**

- ✓ Two complementary techniques are available for **theoretical** modeling of structure formation and evolution in the **nonlinear** regime: **semi-analytic modeling** and **numerical simulations**

- ✓ Concerning to the semi-analytic modeling, two approaches are usual: **spherical collapse** modeling and **hydrodynamical** modeling (*Zeldovich Approximation*, **ZA**)

⇨ Spherical "Top-Hat" Collapse Model

- ✓ Consider the **mean density**, $<\rho>$, at some time. Regions with positive contrasts, $\boldsymbol{\delta > 0}$, are **overdense**, while regions with $\boldsymbol{\delta < 0}$ are **underdense**
- ✓ in the overdense regions, the **self gravity** of the local mass concentration will work **against** the **expansion** of the Univ, i.e., these regions will expand at a progressively **slower** rate compared to the background. Since such slowing down **increases** $\boldsymbol{\delta}$, the gravitational potential of the overdense region will become more and more dominant
- ✓ eventually such a region will **collapse** under its own self gravity and form a **bound system**

- ✓ The details of the above process will depend on the **initial density profile** of the overdense region. The simplest model which one can study analytically is based on the **assumption** that the overdense region is **spherically symmetric**

⇨ Spherical Collapse Model

- ✓ The idea is that the collapsing region **behaves** dynamically like a **small Univ** of slightly higher density. We begin with the Friedmann equation

$$H^2 = (8\pi G/3)\,\rho + \Lambda/3 - kc^2/a^2$$

replacing $\boldsymbol{(\rho + \Lambda/8\pi G) = \rho_T = \rho_0(a_0/a)^3}$ we get

$$\begin{aligned} H^2 &= (8\pi G/3)\,\rho_0\,(a_0/a)^3 - kc^2/a^2 \\ &= (H_0^2/\rho_{crit})\,\rho_0\,(a_0/a)^3 - kc^2/a^2 \\ &= H_0^2\,\Omega_0\,(a_0/a)^3 - kc^2/a^2 \end{aligned}$$

- ✓ Evaluating the above equation at the present epoch ($a = a_0 = 1$)

$$\begin{aligned} &H_0^2 = H_0^2\,\Omega_0 - kc^2 \\ \Rightarrow\ &kc^2 = H_0^2\,(\Omega_0 - 1) \end{aligned}$$

and replacing this result back in the equation

$$\begin{aligned} H^2 &= H_0^2\,\Omega_0/a^3 - [H_0^2\,(\Omega_0 - 1)]/a^2 \\ (da/dt)^2 &= H_0^2\,\Omega_0/a - [H_0^2\,(\Omega_0 - 1)] \\ &= H_0^2\,[(\Omega_0/a) - \Omega_0 + 1)] \end{aligned}$$

⇨ Spherical Collapse Model

✓ Now we replace *a* by **r**, the **radius** of the collapsing region

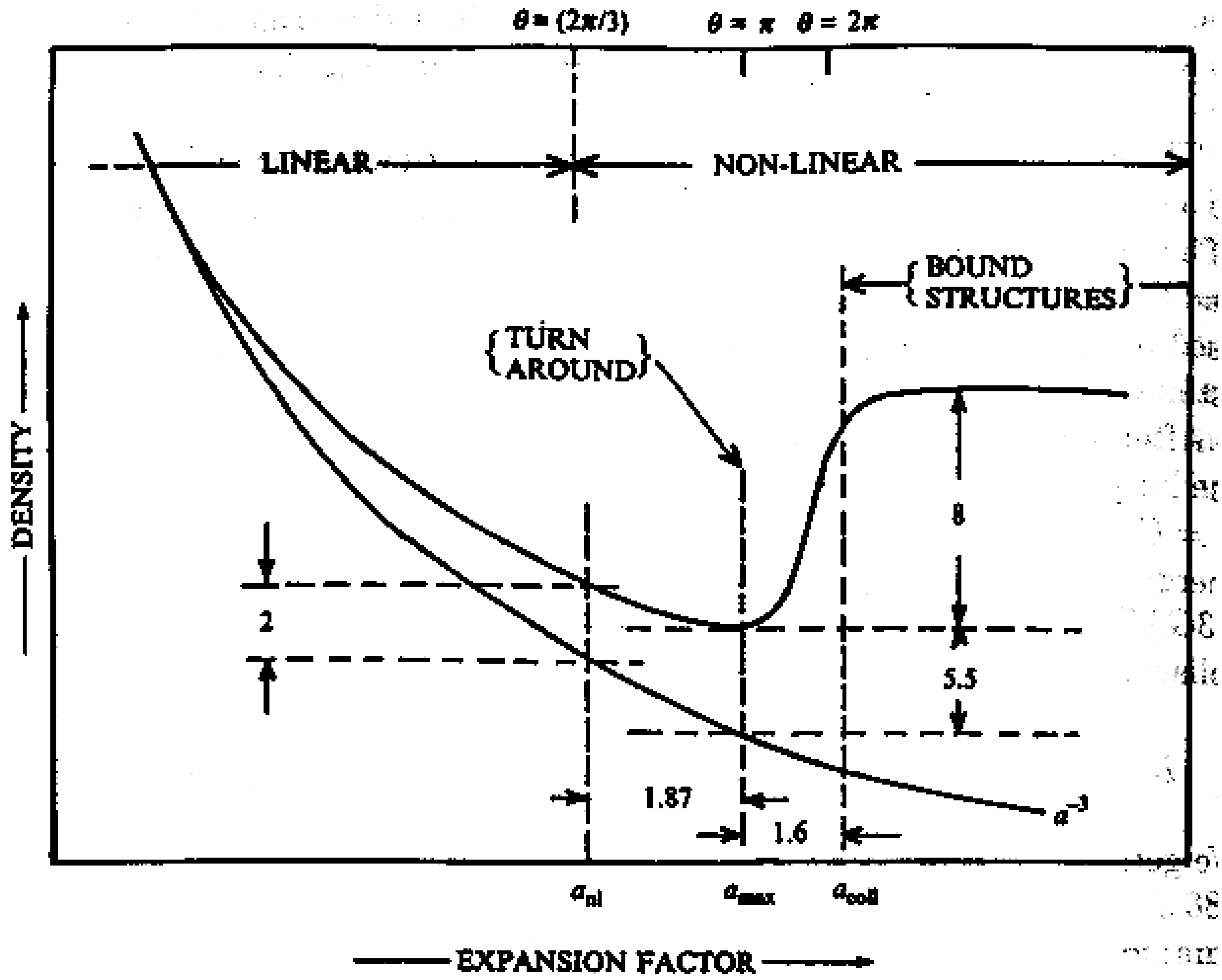
$$(dr/dt)^2 = H_0^2 \, [(\Omega_0 /r) - \Omega_0 + 1]$$

✓ the integration of this equation gives a **cycloidal solution**, which can be conveniently written by the parametric form

$$r = A\,(1 - \cos\theta)$$
$$t = B\,(\theta - \sin\theta)$$
$$A^3 = G\,\mathcal{M}\,B^2$$
$$A = \Omega_0 \,/\, [2(\Omega_0 - 1)]$$
$$B = \Omega_0 \,/\, [2\,H_0\,(\Omega_0 - 1)^{3/2}]$$

✓ This solution implies that the expansion of the perturbation progressively slows down, reaches a **maximum radius** at $\boldsymbol{\theta = \pi}$, called the "*turn-around*", and then **collapse to infinite density** at $\boldsymbol{\theta = 2\pi}$

⇨ Spherical Collapse Model



Padmanabhan 1993, *Structure Formation in the Universe*

⇨ Spherical Collapse Model

✓ Considering the mass inside the collapsing region constant, its **mean density** is

$$\rho(r,t) = \mathcal{M}/V = 3\mathcal{M}/4\pi\, r^3 = 3\mathcal{M}/4\pi\, A^3(1 - \cos\theta)^3$$

✓ while the mean density of the expanding **background** is

$$\rho_{back}(t) = 1/6\pi G\, t^2 = 1\, /\, [6\pi G\, B^2\, (\theta - \sin\theta)^2]$$

✓ Thus, the **density contrast** of the collapsing region is

$$\rho(r,t) = \rho_{back}(t) + \delta\rho = \rho_{back}\, (1 + \delta)$$

$$\Rightarrow\ \rho(r,t)\, /\, \rho_{back}(t) = (1 + \delta) = \frac{3\mathcal{M}\, 6\pi G\, B^2\, (\theta - \sin\theta)^2}{4\pi\, A^3(1 - \cos\theta)^3}$$

$$(1 + \delta) = \frac{G\mathcal{M}\, B^2\, 9\, (\theta - \sin\theta)^2}{2\, A^3(1 - \cos\theta)^3}$$

$$\therefore \qquad \delta = \frac{9\, (\theta - \sin\theta)^2}{2\, (1 - \cos\theta)^3} - 1$$

✓ And so, we can calculate the density contrast at

turn-around:	$\theta = \pi$	$\delta_{ta} = 9\, \pi^2\, /\, 2\, (2^3) - 1 = 9\pi^2/16 - 1 \approx 4.6$	$t_{ta} = \pi B$
collapse:	$\theta = 2\pi$	$\delta_{coll} = 9\, 2\pi\, /\, 0 = \infty$	$t_{coll} = 2\pi B = 2\, t_{ta}$

⇨ Spherical Collapse Model

✓ Interpreted literally, the spherical perturbed region **collapse to a BH**. In practice, it is much more likely to form a **bound object**

✓ Two scenarios are possible

- as the gas cloud collapses, its T increases until internal P gradients become sufficient to **balance** the attractive force of gravity
- during collapse, the cloud **fragments** into sub-units, and then, through the process of ***violent relaxation***#, these sub-units come to a **dynamical equilibrium** under the influence of the gravitational potential of the whole region

✓ In either case, the end result is a system which satisfies the **Virial Theorem**

turn-around: $\mathcal{E} = U \approx -(3/5)\, G\mathcal{M}^2/r_{ta}$

virialization: $U + 2K = 0 \quad \Rightarrow \quad \mathcal{E} = U + K = -K$

$$K = (3/5)\, G\mathcal{M}^2/r_{ta}$$

$$U = -(3/5)\, G\mathcal{M}^2/r_{vir} = -2K = -\mathcal{M}\, v^2$$

$$\therefore \boxed{r_{vir} = r_{ta}/2} \qquad v = (6G\mathcal{M}/5r_{ta})^{1/2}$$

During the collapse there will be large fluctuations in the gravitational potential, in a time scale of the order of the free-fall collapse time, $t_{ff} \sim (G\rho)^{1/2}$. Since the potential is changing with time, individual particles do not follow orbits which conserve the energy (the net effect will be to widen the range of energies available for the particles. This provides a relaxation mechanism for the particles, which operates in a time scale much smaller than the two body relaxation time [Lynden-Bell 1967, MNRAS 136, 101]

⇨ Spherical Collapse Model

✓ We can also estimate the **density of a collapsed object**. Since $\boldsymbol{\rho \propto 1/r^3}$ and $\mathbf{r_{ta} = 2\, r_{vir}}$

$$\rho_{vir} = 8\, \rho_{ta}$$

and $\boldsymbol{\rho_{ta} = \rho_{back}\,(1 + \delta_{ta}) = 5.6\, \rho_{back}}$, and ρ_{back} decreases as

$$a_{vir}/a_{ta} = (t_{vir}/t_{ta})^{2/3} = 2^{2/3}$$

$$\rho_{back,ta}\, /\, \rho_{back,vir} = (a_{vir}/a_{ta})^3 = (t_{vir}/t_{ta})^2 = 2^2 = 4$$

so

$$\boxed{\rho_{vir} = 8 \times 5.6 \times 4\, \rho_{back,vir} \approx 180\, \rho_{back}}$$

✓ Once the system has virialized, its **density** and **size do not change**. Since $\boldsymbol{\rho_{back}} \propto a^{-3}$, the density contrast $\boldsymbol{\delta}$ increases as a^3 for $t > t_{vir}$

⇨ Press-Schechter Formalism

✓ Press and Schechter [1974, ApJ 187, 425] proposed an **analytical formalism** for deriving the **mass function** of **bound objects** in a **CDM** scenario (hierarchical clustering), based in an original **Gaussian distribution of fluctuations**

✓ The idea begins with the assumption that, when the perturbations have developed to amplitude greater than some critical value $\boldsymbol{\delta_c}$, they develop rapidly into bound objects with mass $\mathcal{M}$.

✓ Consider the fluctuation spectrum **P(k)**, and its **rms mass fluctuations** inside a sphere of radius **R**

$$\sigma(\mathcal{M}) = \langle(\delta\mathcal{M}/\mathcal{M})^2\rangle = V/(2\pi)^3 \int d^3k\, W^2(kR)\, |\delta_k|^2$$

$$W(kR) = (3/4\pi R^3) \int_{sphere} d^3\mathbf{x}\, e^{i\mathbf{k}.\mathbf{x}} = 3/(kR)^3\, [\sin(kR) - kR\cos(kR)]$$

✓ If $\boldsymbol{\sigma}$ **(**$\mathcal{M}$**)** is the rms fluctuation amplitude **today**, then at any earlier time it was given by

$$\sigma(\mathcal{M},t) = D(t)\, \sigma(\mathcal{M})$$

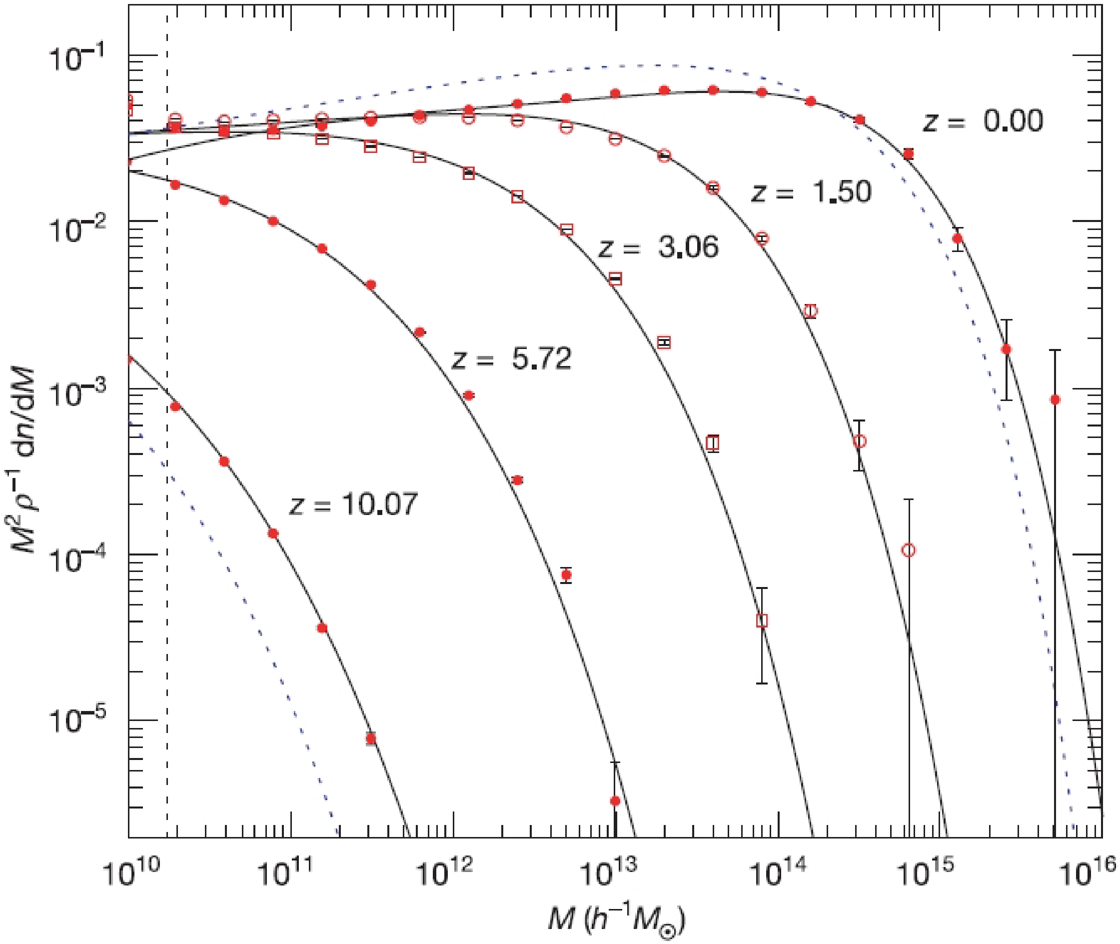
✓ The Gaussian distribution can be recovered by the **error function**, and so, the **fraction of collapsed halos** (bound objects) **more massive than** $\mathcal{M}$ is

$$F(>\mathcal{M}) = 1 - \mathrm{erf}(\delta_c / \sqrt{2}) \qquad\qquad \mathrm{erf}(x) = 2/\sqrt{\pi} \int_{0\to x} dt\, \exp(-t^2)$$

⇨ Press-Schechter Formalism

✓ And, by integrating the last expression, the density of collapsed halos in the mass range ($\mathcal{M}$, $\mathcal{M}$+d$\mathcal{M}$)

$$dn(\mathcal{M},t) = \sqrt{(2/\pi)}\ \ \langle\rho_0\rangle/\mathcal{M}\ \ d\log\sigma/d\mathcal{M}\ \ \delta_c\ \exp(-\delta_c^{\,2}/2)\ d\mathcal{M}$$



Springel et al. 2005, Nature 435, 03597

⇨ Zel'dovich Approximation

- ✓ A more general approach (**ellipsoidal collapse**), using **Lagrangian** coordinates (also called hydrodynamical approach), was presented by Zel'dovich [1970, A&A 5, 84]
- ✓ The proposal of Zel'dovich was also intended to include a scenario with HDM

- ✓ The starting point is the result from the **linear theory** for the growth or small **perturbations**, expressed as a relation between the *Eulerian* and *Lagrangian* coordinates of the particles

$$\mathbf{r} = a(\mathrm{t})\,\mathbf{q} + \mathrm{b}(\mathrm{t})\,\mathbf{p}(\mathbf{q})$$

  where the first term describes the **uniform expansion** of the background and the second one accounts for the **perturbation**

- ✓ Zel'dovich suggested that, while this proposal is in accordance with the linear theory, it may also provide a good **approximate description** of the evolution of density perturbations in the **nonlinear regime**

- ✓ He showed that, in the coordinate system of the principal axes of the ellipsoid, the motion of the particles in comoving coordinates is described by a "**deformation tensor**"

$$\mathrm{D} = \begin{pmatrix} a - \alpha \mathrm{b} & 0 & 0 \\ 0 & a - \beta \mathrm{b} & 0 \\ 0 & 0 & a - \gamma \mathrm{b} \end{pmatrix}$$

⇨ Zel'dovich Approximation

- By conservation of mass, the density **ρ** in the vicinity of any particle is

$$\rho(a-\alpha b)(a-\beta b)\,(a-\gamma b) = \langle\rho\rangle\, a^3$$

- while **α**, **β** and **γ** are functions of the point **q**, ***a*(t)** and **b(t)** are the same for all particles. b(t) is growing **faster** than *a* (t) as a result of gravitational instability.

- If **α** > **β** > **γ** the collapse occur most rapidly along the q_1 axis, and the density becomes infinite when **a(t) – αb(t) = 0**. At this point the ellipsoid has collapsed to a "***pancake***", and the solution breaks down for later times.

- The results of **N-body simulations** have shown that the ZA is quite **remarkably effective** in describing the evolution of the nonlinear stages of the collapse of large scale structures up to the point at which the **caustics** (intersections of trajectories) are formed
- Although density contrasts at this point are already **highly nonlinear**, this model works because the **potentials are smoother** functions than densities and, so, the actual potential may not deviate too much from the linear theory
- At later times, however, ZA predicts the caustics to increasingly **blur out** and the pancakes to **thicken**, while the N-body simulations show that pancakes **remain relatively thin**.

⇨ Adhesion Models

- ✓ The usual improvement of ZA is by the adhesion models. In these models the **particles** are assumed to **stick together** once they enter the region of the caustic
- ✓ The first proposals in this sense were done by Gurbatov et al [1989, MNRAS 236, 385] and Shandarin & Zel'dovich [1989, Rev. Modern Ph. 61, 2]

$$\mathbf{x} = \mathbf{r} / a(\mathrm{t}) \quad \Leftrightarrow \quad \nabla_x = (1/a)\nabla_r$$

$$d\mathbf{x} = (-da/a^2)\ \mathbf{r} + (1/a)\ d\mathbf{r}$$

$$a\, d\mathbf{x}/dt = -H\,\mathbf{r} + d\mathbf{r}/dt$$

$$\mathbf{v}_{pec} = -H\,\mathbf{r} + \mathbf{u}$$

$$\mathbf{x}(\mathbf{q},t) = \mathbf{q} + D(t)\,\mathbf{p}(\mathbf{q})$$

$$\mathbf{r}(\mathbf{q},t) = a(t)\,\mathbf{q} + a(t)\,D(t)\ \mathbf{p}(\mathbf{q})$$

$$\mathbf{u} = d\mathbf{r}/dt = a'\mathbf{q} + a'D\,\mathbf{p} + a\,D'\,\mathbf{p}$$

$$\mathbf{v} = -H\,\mathbf{r} + \mathbf{u} = -(a'/a)[a\,\mathbf{q} + a\,D\,\mathbf{p}] + \mathbf{u}$$

$$= \cancel{-a'\mathbf{q}} - \cancel{a'D\,\mathbf{p}} + \cancel{a'\mathbf{q}} + \cancel{a'D\,\mathbf{p}} + a\,D'\,\mathbf{p}$$

$$\therefore \quad \mathbf{v} = a\,D'\,\mathbf{p}(\mathbf{q})$$

$$d\mathbf{v}/dt = a'\,D'\,\mathbf{p} + a\,D''\mathbf{p}$$

$$= (a'\,D' + a\,D'')\,\mathbf{v} / a\,D'$$

$$= (a'/a)\,\mathbf{v} + (D''/D')\,\mathbf{v}$$

⇨ Adhesion Models

$$d\rho/dt + \rho\, \nabla_x \cdot \mathbf{u} = 0$$
$$d\rho/dt + \rho\, \nabla_x \cdot (\mathbf{v} + H\, \mathbf{r}) = 0$$
$$d\rho/dt + \rho\, \nabla_x \cdot \mathbf{v} + \rho\, H\, \nabla_x \cdot \mathbf{r} = 0$$
$$d\rho/dt + (1/a)\, \nabla_r \cdot \rho\mathbf{v} + 3\, H\, \rho = 0$$

$$d\mathbf{u}/dt = -\nabla_x \Phi$$
$$d/dt\, (\mathbf{v} + H\, \mathbf{r}) = -(1/a)\, \nabla_r \Phi$$
$$d\mathbf{v}/dt + H\, \mathbf{u} = -(1/a)\, \nabla_r \Phi$$

$$\nabla_x^2 \Phi = 4\pi G\, (\rho - \langle\rho\rangle)$$
$$(1/a^2)\nabla_r^2 \Phi = 4\pi G\, (\rho - \langle\rho\rangle)$$
$$\nabla_r^2 \Phi = 4\pi G\, a^2\, (\rho - \langle\rho\rangle)$$

$$\boldsymbol{\mathsf{v}} \equiv \mathbf{v} / a D'$$
$$\eta \equiv a^3 \rho$$

✓ In terms of the new variables

$$d\eta/dD + \nabla_r \cdot \eta\, \boldsymbol{\mathsf{v}} = 0$$

$$d\boldsymbol{\mathsf{v}}/dD + (\boldsymbol{\mathsf{v}}\, \nabla_r)\, \boldsymbol{\mathsf{v}} = \mathbf{0}$$

✓ and including *ad-hoc* a term for a virtual **viscosity***

$$d\boldsymbol{\mathsf{v}}/dD + (\boldsymbol{\mathsf{v}}\, \nabla_r)\, \boldsymbol{\mathsf{v}} = \nu\, \nabla_r^2\, \boldsymbol{\mathsf{v}}$$

✓ that makes it similar to the **Burgers' Equation.** Note that **ν** is very small (in fact, **ν** → **0**)

* An example of a temptative explanation for the virtual viscosity is given in Ribeiro & Peixoto [2005, astro-ph/0502580]

⇨ Spin Parameter

- ✓ In a broad manner one can consider 2 basic types of galactic systems: **discs** (spirals) and **spheroids** (ellipticals)

- ✓ It is useful to define the **spin parameter** of a galaxy, that is related to its angular momentum, **L**

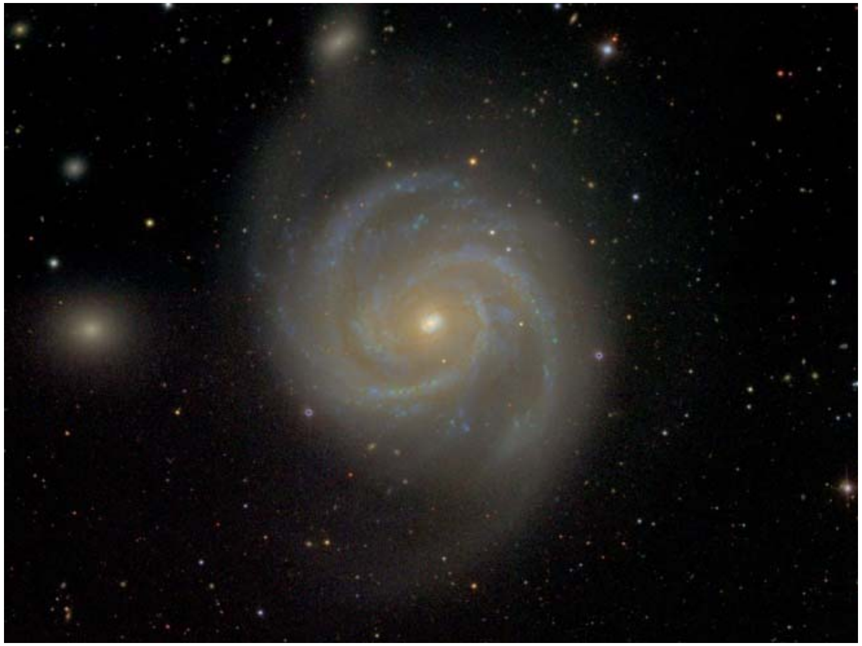
$$\lambda = L U^{1/2} / G \mathcal{M}^{5/2}$$

and indicates how much of the galaxy gravitational **support** is given **by rotation**. Disc galaxies have λ between **0.4 and 0.5**, while spheroids have **λ** ~ **0.05**

⇨ Formation of Disc Galaxies

- ✓ Thus, **disc galaxies** owe their equilibrium to **rotational support** [typical rotation velocities range from (200-300) km/s]. **SF is inefficient** and the left-over gas forms the disc.

- ✓At present the most popular idea is that galaxies **acquire** their **angular momentum** through the **tidal torques** due to their **neighbors**
- ✓ However, N-body simulations show that tidal torques are able to give, at best, **only 5-10%** of the L needed for the rotational support.
- ✓ If the disc galaxies contain **massive DM halos**, the increase of **binding energy** during **dissipative collapse** of the gas (since mass and L remain the same) will **increase the spin parameter** enough for given the pressure support we observe.



⇨ Formation of Spheroidal Galaxies

- ✓ Elliptical galaxies are systems of stars which are **supported** against gravity by their **random motions**

- ✓ The simplest model for the formation of E is the **monolithic** one [Eggen, Lynden-Bell & Sandage 1962, ApJ 136, 748]. In this model, E form at **high z** by **dissipationless collapse** and **SB**. Stars can form rapidly in dense protogalaxies even before the turn-around radius is reached, and then **relax violently** to form spheroidal galaxies. The major problem with this model is it **predicts** that the ellipticity is due to **flattening by rotation**, while observations suggest that E are **triaxial** and owe their shape to **anisotropic velocity dispersions**.

⇨ Formation of Spheroidal Galaxies

✓ One widely discussed idea is that **all galaxies were originally formed as S**, and **E arise from the merger** of disc systems.

✓ The positive features of this hypothesis are:
- galaxy **mergers** are actually **seen to occur** (and probably were more frequent in the past);
- even some smooth light profile E show **signs** that they have experienced **mergers** (shells, tidal tails, inclined gas discs, etc);
- **numerical simulations** of mergers between galaxies of comparable size show that the resultant systems usually resemble E (since merging discs will have their **spins randomly oriented** with respect to each other, the resultant spin can be considerably smaller than the originals);

✓ The difficulties of this hypothesis, on the other hand, are the following:
- E are more abundant in rich clusters, but the **vel dispersions** in these environments (~ 1000 km/s) make mergers rate improbable
- in a dissipationless merger, energy per unit mass is conserved, but observations show that E have **much deeper potential** than typical S
- some E have higher **phase space densities** in the core than found in any disc galaxy (improbably increased in a dissipationless merger)

⇨ Formation of Spheroidal Galaxies

- ✓ More plausible scenarios emerge with **refining** and **modifying** the original **merger hypothesis**

- ✓ **Hierarchical clustering** can solve the first problem: velocity dispersion increases with mass and so, in **small subclusters** the random bulk velocities of the galaxies will be smaller and mergers can take place more easily – these subclusters later combine hierarchically to form rich clusters

- ✓ The presence of **extended dark haloes** leads to significant dynamical friction on the galaxies as they move through each other's halo, enabling the merger. As the discs merge, they can **become more strongly bound by transferring the energy to the halo**. Dynamical friction also contributes to lose orbital angular momentum

- ✓ Proper account for the gas during their merger can solve issue of high phase space densities in the cores: **gas can dissipate energy and sink to the centre**

- ✓ Another interesting picture is based on mergers but with progenitors that are **subgalactic clumps** and not full-blown discs.

References

Books:

- T. Padmanabhan 1993, *Structure Formation in the Universe*, Cambridge Univ. Press
- M. Longair 1998, *Galaxy Formation*, A&A Library – Springer
- P. Coles & F. Lucchin 1995, *Cosmology: The Origin and Evolution of Cosmic Structure*, Wiley

Additional papers:

- B.J.T. Jones et al. 2004, Rev. Mod. Ph. 76, 1211
- B. Ciardi 2004, arXiv 0409018