Definition of science

Based on Lucio Russo discussion (2003 - *The forgotten revolution: how science was born in 300BC and why it had to be reborn*)

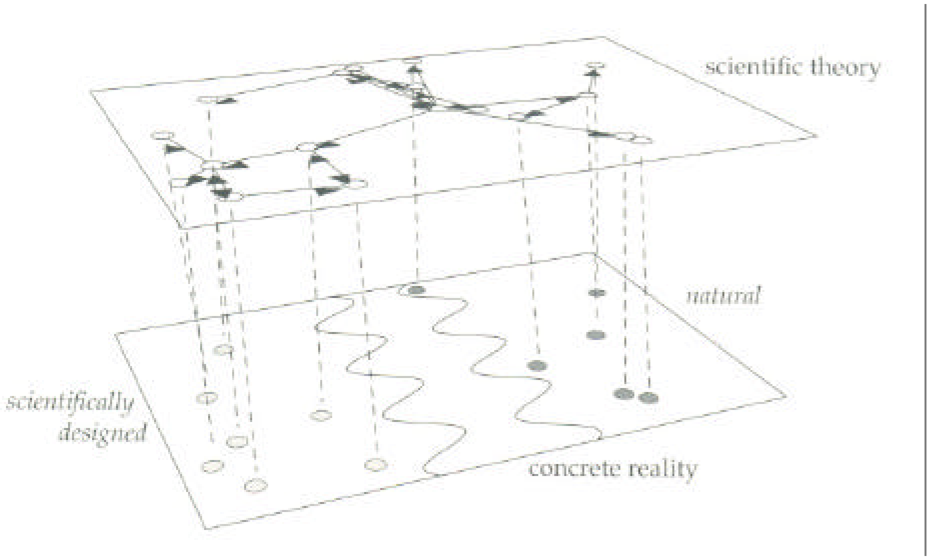
A "Classical" or standard description of science:

1- Natural (empirical) science:
A coarsely ENCYCLOPEDIC organization of knowledge that characterizes a particular object

- A container in which are to be placed all the true statements describing the specific object chosen

2- Pure (theoretical or higher level) science:
With no correspondence rules for application to real world – Ex. contemporary Mathematics

Russo: "Scientific theories, even if created for the purpose of describing natural phenomena, are able to **enlarge themselves by means of the deductive method**, and as a consequence they usually develop into models of areas of technological activity"

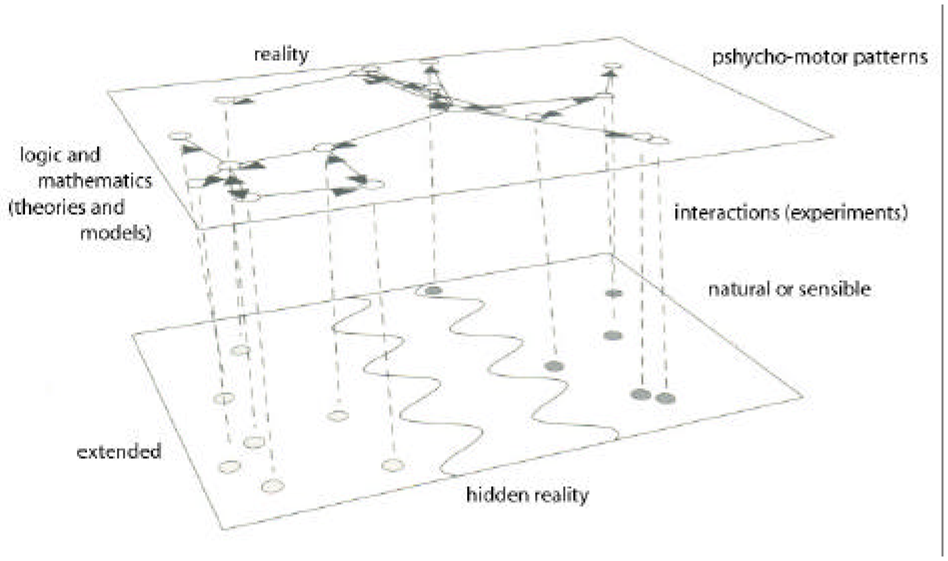


Definition of Intelligence

Adaptation of "classical" view of science based on the definition of intelligence by Psychologist Jean Piaget (1949 *Introduction à l'épistémologie génetique*)

Piaget: "The intelligence is the **integration** of the **action** of the **subject** on the **object**"

- **Subject** = **conscience**, source of **interaction** and **integration**
- **Object** = **hidden reality** – assumed, but not directly observable
- **Actions** = all possible interactions with reality ➔ our only source of information about hidden reality ➔ related in the brain with psycho-motor patterns ➔ form ideas and concepts
- **Integration** = arrays of interconnections in brain related with psycho-motor patterns; allows new connections ➔ produce logic and mathematics

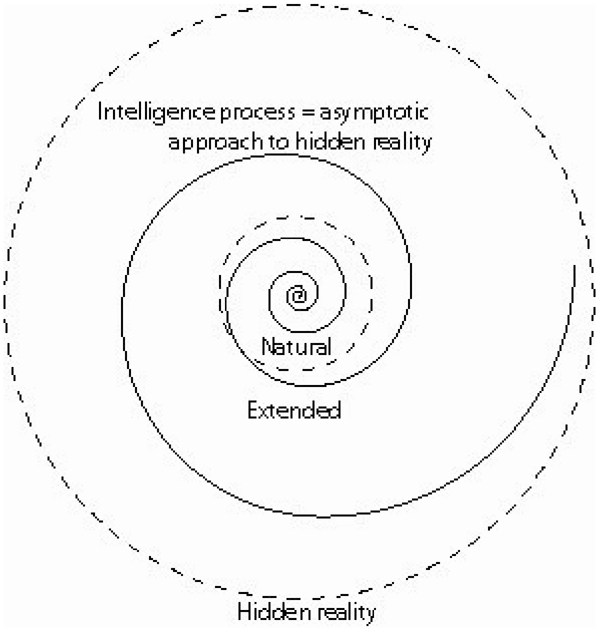


Consequences:

- **Reality** is an image in the brain (subject) which is build (integration) through our interactions (actions) with "hidden reality" (object)
- **Observation** (**experiment**) is never direct = ensemble of interactions possible on hidden reality ➔ We cannot separate the subject from the object
- ***Through logic and mathematics, we increase our possibility of interactions ➔ extending our "knowledge" of hidden reality – verified by new experiences***

The intelligence is a process – ***the adaptation of subject to "hidden reality"***

➔ It allows better adaptation to hidden reality = increasing our chances of survival



Note: asymptotic approach to hidden reality ➔ open system, not bounded, ***there is no limit to adaptation***

Science is a **social and organized** activity related with the development of intelligence

- Its goal is to produce a better integration, enlarging and optimizing our interactions with hidden reality
- This is a typical anthropic activity, which defines and characterizes human

What is the hidden (veiled) reality? – see Bernard D'Espagnat 2006 *On Physics and Philosophy* or 2003 *Veiled Reality: An Analysis of Present-Day Quantum Mechanical Concepts*

We know about nature only through our interactions – **physics study the nature of interactions between "matter" not matter itself** – through our study of these interactions we form an image (model) of reality – therefore, in some way nature is hidden or veiled – **reality is also veiled by the true goal of intelligence, which is to give us an edge for survival** ***– what is the absolute nature of the hidden reality is not the relevant question, our survival is*** – the "truth" of our model of reality is determined by the efficiency of our actions on hidden reality

The theory of **physical mathematics** must be understood as describing and extending the sets of possible interactions on hidden reality

Numerical system – theory of number

The numerical system was developed to increase the possibilities of actions = extending our possible interactions with hidden reality

Normal social life + experimental physics + engineering ➔ deal only with **integers** (**natural numbers**) and **rational numbers**

In theoretical physics **real** and **complex** numbers were invented because they provide more efficient models ➔ increase our set of possible actions on hidden reality

Scalar = **positive integer** $\mathbb{I}$ (*natural number*) – describes some specific quantity (taken as quality = magnitude) about an **event** (interaction with hidden reality)

Examples: Number of distinguishable objects ("with same qualities"), its apparent length or a combination, like area and volume, but also its mass or weight, or the time lapse for the application of an action on the object, etc.

Basic logical operations if $A, B \in \mathbb{I}$ then

1- **Equivalence relations**
   - **Reflexivity (identity)**: A = A
   - **Symmetry**: if A = B then B = A
   - **Transitivity**: if A = B and C = B then A = C

2- **Proportionality** $A \propto B$ which extend logically to identity $(q = 1)$
   - $A = qB \Rightarrow \frac{A}{B} = q$, where q is a constant

Rational number $\mathbb{Q}$: $q = \frac{A}{B}$: there exist two integers q and r such that $A = qB + r$

In physics, proportionality is extremely important ➔ defines some constant (**physical constant**) characteristics of specific interactions with hidden reality

- Ex. mass–energy relation $\frac{E}{m} = c^2$, where c is the velocity of light
- Or quanta of light related to frequency $\frac{E}{n} = h$, where h is Planck constant

Note that both constants define a basic "scale'' of matter (the magnitude of interaction): an atom absorb or emit light if the difference in frequency is equivalent to a difference in energy $\Delta n\, h = \Delta E$ and similarly the fusion of atom produce a difference of mass equivalent to a difference in energy $\Delta mc^2 = \Delta E$

Mathematical formalism

A **group** $(G,\circ)$ is an algebraic structure consisting of a **set** G together with an **operation** $\circ$ that combines any two of its elements to form a third element

To qualify as a group, the set and the operation must satisfy a few conditions called **group axioms**, namely ***closure***, ***associativity***, ***identity*** and ***invertibility***

The symbol $\circ$ (circle) describes the possible **operation** (**action, relation or transformation**) on a **set** of mathematical entities – ex. **binary operation** $(a \circ b)$

- **Closure** : if $x, y \in \mathbb{I}$ then $x \circ y = z \in \mathbb{I}$

Two basic operations defined for set $\mathbb{I}$: $+$ and $\times$

- **Addition**: $x + y = z$
- **Multiplication**: $x \times y = z$

Basic properties of operations:

- **Commutative** $x + y = y + x$
- **Associative** $x \times (y \times z) = (x \times y) \times z$

For $\times$ exists an **identity element** $\mathbb{e} = 1 \in \mathbb{I}$ such that $\mathbb{e} \times x = x$

$\mathbb{I}$ = **Semigroup** *under multiplication* $(\mathbb{Z}, \times)$ or $\mathbb{Z}_{\times}$

For addition, there is no identity $\mathbb{e} \in \mathbb{I}$ such that $\mathbb{e} + x = x$

$\Rightarrow$ **To extend the addition operation we must invent a new number** $\mathbb{e} = 0 \notin \mathbb{I}$

$\mathbb{I}$ = **Semigroup** *under addition* $(\mathbb{Z}, +)$ or $\mathbb{Z}_{+}$

The process of extension of action continues through the notion of **inverse**:

Neither addition nor multiplication as an **inverse** within $\mathbb{I}$: $x \circ k = \mathbb{e}$, with $k \in \mathbb{I}$

For addition we must invent **negative integers**: $x + (-x) = 0$

Within the inclusion of negative integers and zero, $\mathbb{Z}$, the equation $p + x = q$ has a unique integer solution $x(\equiv q - p)$ for every pair of integers p, q ➔ $\mathbb{Z}$ forms a **Group** *under addition*

The fact that addition is commutative makes $\mathbb{Z}$ a **commutative** or **abelian group**

The combined operations of $+$ with $\mathbb{e}=0$ and $\times$ with associativity and $\mathbb{e}=1$ characterize $\mathbb{Z}$ as a **ring**, or **commutative ring** since $\times$ is commutative

A **ring** is an algebraic structure consisting of a set together with two binary operations **addition** and **multiplication**, where the set is an **abelian group under addition** and a **monoid under multiplication**, such that multiplication distributes over addition

$$a\times(x+y)=(a\times x)+(a\times y)$$

The **ring axioms** require that ***addition is commutative***, ***addition and multiplication are associative***, ***multiplication distributes over addition***, ***each element in the set has an additive inverse***, and ***there exists an additive identity***

To introduce an **inverse for multiplication** we must introduce **rational numbers**:

such that $\frac{p}{q}$ where $p,q\in\mathbb{I}$

Rational number $\mathbb{Q}$ has one important property = ***they can always be reduced to a form in which the integers*** p,q ***have no common factors*** (**prime numbers**)

With $+$ and $\times$ having well defined inverses (except for 0, which is undefined) and the **distributive law** $a\times(x+y)=(a\times x)+(a\times y)$, the **rational number** $\mathbb{Q}$ **form a field**

A **field** is an algebraic structure with notions of ***addition***, ***subtraction***, ***multiplication***, and ***division***, satisfying certain axioms

Any field may be used as the scalars for a ***vector space***

Every field is a ring, but not every ring is a field - the most important difference is that fields allow for division (though not division by zero), while a ring need not possess multiplicative inverses - also, the multiplication operation in a field is required to be commutative

The importance of groups in mathematical physics

Groups share a fundamental kinship with the notion of symmetry in geometry

A **symmetry group** encodes the symmetry features of a geometrical object $\Rightarrow$ it consists of the set of **transformations** that leave the object unchanged, and the **operation of combining two such transformations** by performing one after the other

Ex. **Symmetric group** S_N **of permutations** of N objects

Symmetries link with invariance laws in mathematical description of events $\Leftrightarrow$ conservation laws in physics – **groups of isometries** (preserve distances): they are the products of **reflections**, **translations** and **rotations**

- **Translational invariance** = the dynamical laws (set of interactions) that describe the evolution in time of a system are independent of the choice of origin of the coordinate system $\Leftrightarrow$ law of ***conservation of momentum***
- **Rotational invariance** $\Leftrightarrow$ law of ***conservation of angular momentum***
- **Reflexion invariance (independence on initial time**) $\Leftrightarrow$ law of ***conservation of energy***

In solids, where full symmetries are not present discrete rotational and translational symmetries of lattices define special physical properties of the solids $\Leftrightarrow$ **Finite groups** describes symmetry of geometric structures of molecules (interactions between molecules) in crystal

Continuous Lie groups (continuous transformation groups)

- ***space-time symmetries*** = principle of equivalence of Einstein, the ***curvature of space-time*** $\Leftrightarrow$ ***gravity***
- ***conservation of electric charges*** $\Leftrightarrow$ ***fundamental interactions of quarks and leptons***

Relation not that difficult (or obscure) to understand, when we remember that what the mathematics (ex. geometry of space-time) really describe is the set of interactions (action or dynamics) possible on hidden reality, not reality itself

In QM symmetries is extended to the structure of matter
Ex. systems of identical particles have special properties

- **Bosons** (integer spin) ➔ state of system must be symmetric under permutation of particles
- **Fermions** (1/2 spin) – state of system must be antisymmetric under permutation of particles ➔ *basis of Pauli exclusion principle*

In QM one cannot separate matter from space-time:
➔ hidden reality in QM $\Leftrightarrow$ **matter-space-time**

Further extension of operations = irrational and complex numbers

Rational numbers cannot solve algebraic (polynomial) equations, or carry out the limiting operations of calculus – must invent **irrational numbers**

Ex. $x^2 - 2 = 0 \Rightarrow x = \sqrt{2} \notin \mathbb{Q}$

Also is p is a **prime number** then $\sqrt{p} \notin \mathbb{Q}$

Real algebraic numbers $\mathbb{R}$ extend operations defined as obtaining the real roots of polynomials of any degree with integer coefficients

$\mathbb{R}$ forms a **field**

➔ The roots of a polynomial with rational coefficients can be expressed as roots of a polynomial with integer coefficients

But some algebraic equations do not seem to have real roots – must invent **imaginary numbers**

Ex. $x^2 + 1 = 0 \Rightarrow x = \sqrt{-1} \notin \mathbb{R}$

This can be solved by inventing **imaginary unit** $i \equiv \sqrt{-1}$ such that the roots are given by $x = \pm i$

Complex numbers $\mathbb{C}$ introduced as ordered pairs $(x, y) \sim x + iy = z$ of real numbers x, y

Can be represented as points in a plane (the **complex plane**) with **magnitude** $|x+iy| = \sqrt{x^2 + y^2}$ and in view of the identity $e^{i\theta} = \cos\theta + i\sin\theta$ we can also write $x + iy = re^{i\theta}$ with $r = |x+iy|$ and $\tan\theta = \frac{y}{x}$

$\arg z \equiv \theta$ for $z \neq 0$ (defined modulo 2π ➔ adding any integer multiple of 2π to $\arg z$ does not change the complex number z ➔ $e^{2\pi i} = 1$; this is **Euler equation**

Fundamental theorem of algebra
Every polynomial has at least one root in the complex plane ➔ every polynomial of degree n has exactly n roots in the complex plane when these roots are suitably counted

Complex numbers $\mathbb{C}$ form a field

Other extensions in mathematical physics

Infinite series

Like $\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \cdots$ or $\prod_{n}^{\infty} x_n = x_1 x_2 x_3 \cdots$

Central to the study of exact and approximate solutions to differential equations arising in every branch of physics

Many functions are defined only through series and it is important to understand the convergence properties of these series both for theoretical analysis and for approximate evaluation of the functions

- **Absolute convergence** of an infinite series is necessary and sufficient to allow the terms of a series to be rearranged arbitrarily without changing the sum of the series (invariance in permutation or order of terms of the sum)

Infinite sequences of functions

Uniform convergence – guarantees that properties like **continuity** and **differentiability** of the functions of the sequence are shared by the limit function

Weak convergence – defined in terms of the sequence of numbers generated by integrating each function of the sequence over a region with functions from a class of smooth functions; for example **Dirac d - function and derivatives**

Infinite series of functions or **Power series**

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

where a_n real or complex numbers and z is complex variable – central to theory of functions of complex variable

- A power series converges **absolutely** and **uniformly** inside a circle in the complex plane (circle of convergence) with convergence on circle of convergence depending on particularity of series

Divergent series – semiconvergent or **asymptotic series –** used to determine asymptotic behavior and approximate asymptotic values of a function (ex. Laplace methods)

Beyond the sequences of series generated by the mathematical functions that occur in solutions to differential equations of physics there are sequences generated by dynamical systems (equations of motion) – ***can be viewed as iterated maps of the coordinate space of the system into itself***

Asymptotic behavior exhibits new phenomena beyond simple convergence or divergence

- Sequences that converge to periodic limit cycle
- Or that diverge in such a way that the points in the sequence are dense in some region in a coordinate space

Ex = **logistic map *(causal system)***

$T_l : x \to x_l = l\,x(1-x)$

Starting from a generic point x_0 in the interval $0 < x_0 < 1$, generates a sequence of points $\{x_n\}$ with $x_{n+1} = l\,x_n(1-x_n)$ for $(0 < l < 4)$

This map describes **a discrete-time dynamical system** ➔ takes each of the possible initial states of the system into its successor

Examples:

- successive interactions of a particle orbit with a fixed plane
- population counts of various species in an ecosystem at definite intervals of time

Provides illustration of the phenomena of **period doubling** and **transition to chaos → evolution of dynamical system fail to show simple causal relation with initial condition**

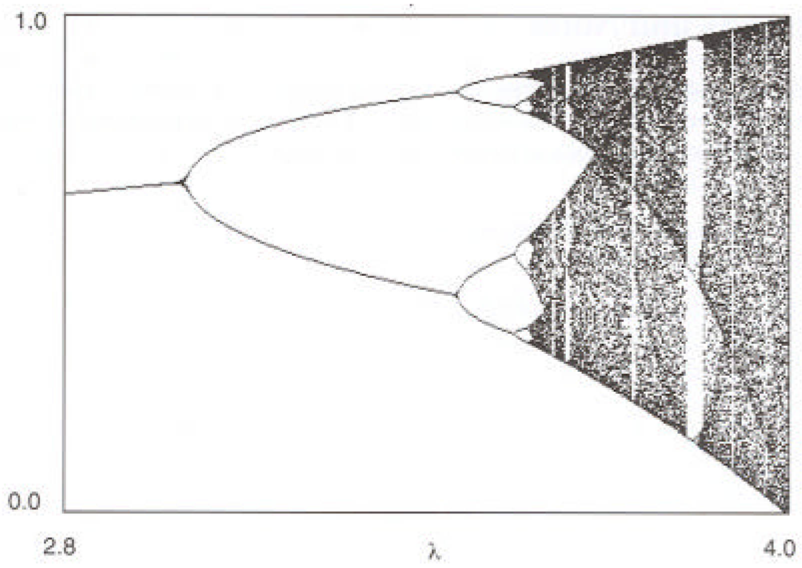


Figure 1.1: Iterates of the map (1.A2) for λ between 2.8 and 4.0. Shown are 100 iterates of the map after first iterating 200 times to let the dependence on the initial point die down.

For the sequence $x_{n+1} = \lambda x_n \left(1 - x_n\right)$

- For $\lambda < 1$, the sequence converge to 0
- For $\lambda > 1$, do not converge to 0
- For $\lambda > 3$, sequence oscillates between two different solutions (period 2)
- For $\lambda = 1 + \sqrt{6} \cong 3.4495$, bifurcation into stable cycles of length 4 (period doubling)
- The bifurcation continue after shorter and shorter intervals of λ, until reach $\lambda_c \cong 3.56994$, where the sequence becomes chaotic
- With highlands of periodicity and return to chaos

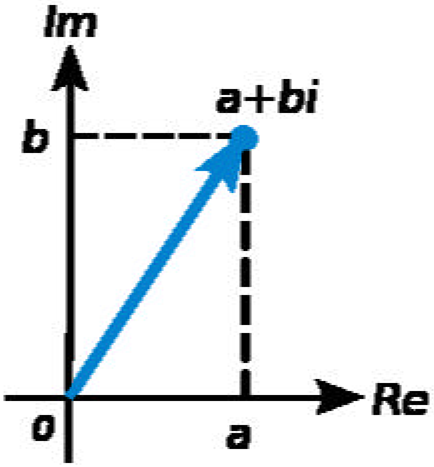
Complex numbers

Most important extension of mathematical structures

Are necessary to solve equations of the kind $x^2 = -1$

First used by Girolamo Cardano (1501-1576) and developed further by Abraham de Moivre (1667 – 1754), Leonhard Euler (1707-1783), Jean Robert Argand (1768-1822), Carl Friedrich Gauss (1777-1865), Augustin Louis Cauchy (1789-1857), Karl Weierstrass (1815-1897) and Bernhard Riemann (1826-1866), to name a few

The solution of $x^2 = -1$ is an **imaginary** or **complex number** $i^2 = -1$



Complex because it is made of two parts: $z = (x, y)$ where $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$

Therefore $i = (0,1)$ and any pure imaginary number $y = (0, y)$ while real numbers are equal to $x = (x, 0)$

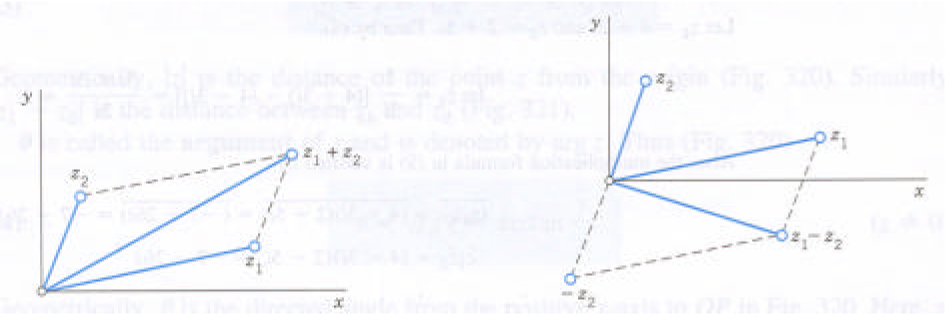
Extend the real space $\mathbb{R}$ (continuous axe) with an imaginary continuous axe $\mathbb{I}$, the two forming the **complex plane** $\mathbb{C}$ - or **Argand Diagram**

NOTE: the form looks like a vector, but it is not a vector in terms of operations **– *the imaginary axe is not a spatial-axe***

Possible operations

In practice we can write $z = x + yi$

Addition similar to vector addition:



If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ then $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$

Therefore $z = x + yi = (x, 0) + (0, y)$, which has the same form as a vector addition

But the rule for **multiplication** is similar to that real numbers applying the distribution of multiplication on addition:

$$z_1 z_2 = (x_1 + y_1 i)(x_2 + y_2 i) = x_1 x_2 + y_1 y_2 i^2 + x_1 y_2 i + x_2 y_1 i = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i$$

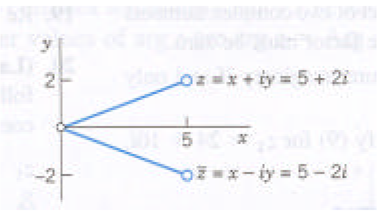
Another fundamental difference is that **division** is consequently possible (not the case of vector): $z = \frac{z_1}{z_2} \Rightarrow z_1 = z z_2 = x_1 + y_1 i$ for $z_2 \neq 0$

Applying the rule of multiplication: $\begin{aligned} x_2 x - y_2 y &= x_1 \\ y_2 x + x_2 y &= y_1 \end{aligned}$

Multiply first equation by x_2 and second by y_2 and summing to eliminate the imaginary term of z finding $x = \frac{x_2 x_1 + y_2 y_1}{x_2^2 + y_2^2}$

Similarly we can eliminate the real term of z finding $y = \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$

We define the **complex conjugate** of $z = x + yi$ as $\overline{z} = x - yi$



Therefore $z\overline{z} = x^2 + y^2$ this is a real number

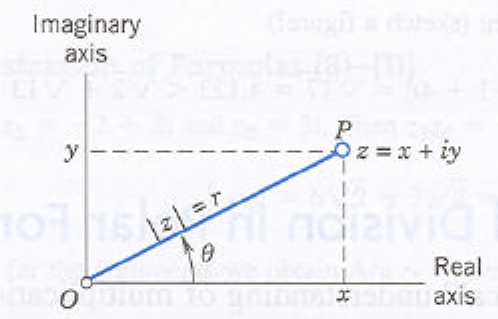
And $z = \dfrac{z_1}{z_2} = \dfrac{z_1}{z_2} \cdot \dfrac{\overline{z}_2}{\overline{z}_2}$

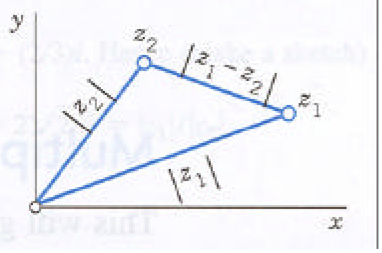
The four fundamental operations defined in this way, the complex numbers set follow the standard commutative, associative and distributive laws with identities and inverses – it forms a field

Once defined, the complex conjugate it is easy to find that

- $\operatorname{Re} z = \dfrac{1}{2}\left(z + \overline{z}\right)$ while $\operatorname{Im} z = \dfrac{1}{2i}\left(z - \overline{z}\right)$
- If z is real, then $z = \overline{z}$
- And we have the following rules $\overline{\left(z_1 + z_2\right)} = \overline{z}_1 + \overline{z}_2$ and $\overline{\left(z_1 z_2\right)} = \overline{z}_1 \overline{z}_2$

The fact that we can represent a complex number in a plane suggests a **polar form** representation





$x = r\cos\theta$ and $y = r\sin\theta$ such that $z = r\left(\cos\theta + i\sin\theta\right)$

The **absolute value** or **modulus** $|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\overline{z}}$

- Geometrically $|z|$ is the distance of the point z from the origin, whence $|z_1 - z_2|$ is the distance between z_1 and z_2

The angle θ is the **argument** defined as $\theta = \arg z = \arctan\frac{y}{x}$

Where the angles are measured in radians and are positive in the counterclockwise sense

Since sine and cosine are periodic over 2π, we define the **principal value** $\operatorname{Arg} z$ has $-\pi < \operatorname{Arg} \leq \pi$ such that $\arg z = \operatorname{Arg} z \pm 2n\pi$ with $n = \pm 1, \pm 2, \ldots$

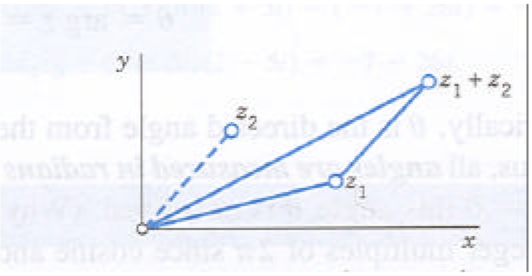
- For $z = 0$, θ is undefined
- positive real number have $\operatorname{Arg} z = 0$
- negative real number have $\operatorname{Arg} z = \pi$

Inequalities such as $x_1 < x_2$ makes sense for real numbers, but not for complex numbers since

- *there is no natural way of ordering complex numbers*

Inequalities are important for absolute values (which are real), for example
$|z_1| < |z_2|$ means z_1 closer to origin than z_2

Triangle inequality $|z_1 + z_2| \leq |z_1| + |z_2|$



By induction we obtain the **generalized triangle inequality**

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

The absolute value of a sum cannot exceed the sum of the absolute values of the terms

Operations in polar form

Addition and subtraction are straightforward, since they are similar to vectors, however, this is not the case of multiplication and division

If $z_1 = r_1\left(\cos\theta_1 + i\sin\theta_1\right)$ and $z_2 = r_2\left(\cos\theta_2 + i\sin\theta_2\right)$ the application of the multiplication rule $z_1 z_2 = r_1 r_2\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$

Using the trigonometric identities $\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = \cos(\theta_1+\theta_2)$ and $\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 = \sin(\theta_1+\theta_2)$

$$\Rightarrow z_1 z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin\left(\theta_1+\theta_2\right)\right]$$

Taking the absolute value on both sides $\left|z_1 z_2\right| = \left|z_1\right|\left|z_2\right|$
Taking the argument $\arg\left(z_1 z_2\right) = \arg z_1 + \arg z_2$ up to multiple of 2π

These basic rules simplifies what we should expect for division
$\left|\dfrac{z_1}{z_2}\right| = \dfrac{\left|z_1\right|}{\left|z_2\right|}$ and $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

Therefore $\Rightarrow \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin\left(\theta_1-\theta_2\right)\right]$

Putting $z_1 = z_2 = z$ and from induction we get **de Moivre's formula** for integer powers of $z \Rightarrow z^n = r^n\left[\cos(n\theta) + i\sin\left(n\theta\right)\right]$

Similarly for $z_1 = 1$ and $z_2 = z^n$ we have **de Moivre's formula** for $n = -1, -2, \ldots$ and in general $\Rightarrow z^p = r^p\left[\cos(p\theta) + i\sin\left(p\theta\right)\right]$ for any real number p

For $|z| = r = 1$, $z = \left[\cos(\theta) + i\sin\left(\theta\right)\right]^n = \cos(n\theta) + i\sin\left(n\theta\right)$

Very useful to express $\cos n\theta$ and $\sin n\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$

Example: for $n = 2$,
$\left[\cos(\theta) + i\sin\left(\theta\right)\right]^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta = \cos(2\theta) + i\sin\left(2\theta\right)$
Comparison suggests that $\cos^2\theta - \sin^2\theta = \cos(2\theta)$ and $2\cos\theta\sin\theta = \sin\left(2\theta\right)$

Roots

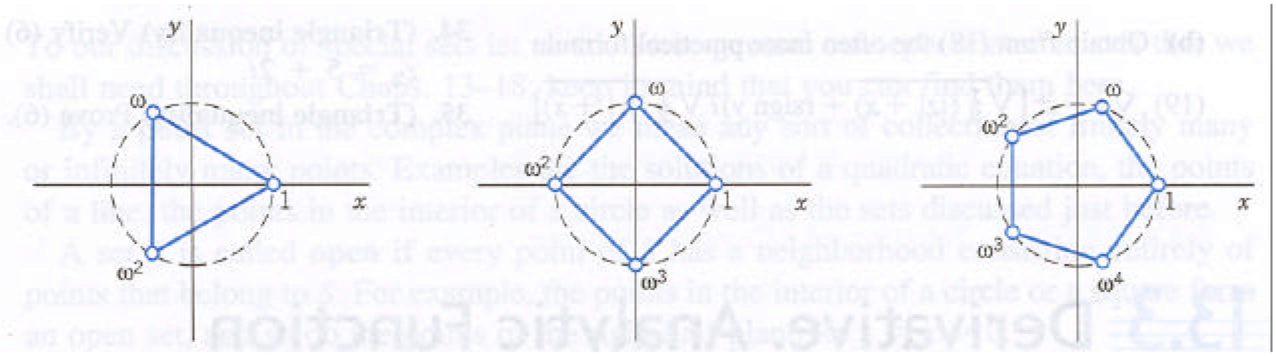
Once defined, the power function yields that if $z = w^n$ then to a given $z \neq 0$ must correspond precisely n distinct values of w called the **nth roots** of z: $w = \sqrt[n]{z}$

Writing $z = r(\cos q + i\sin q)$ and $w = R(\cos f + i\sin f)$, by de Moivre’s formula, the equation $w = \sqrt[n]{z}$ becomes $w^n = R^n(\cos nf + i\sin nf) = r(\cos q + i\sin q)$

Where $R = \sqrt[n]{r}$ and $nf = q + 2kp$ or $f = \frac{q}{n} + \frac{2kp}{n}$ and k is an integer $k = 0,1,2,\ldots,n-1$, which yields n distinct values

Therefore $\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{q+2kp}{n} + i\sin\frac{q+2kp}{n}\right)$

These values lies on a circle of radius $\sqrt[n]{r}$ with center at origin and constitute the vertices of a regular polygon of n-sides (principal value of $\arg z$ and $k = 0$ is the principal value of $w = \sqrt[n]{z}$



Taking $z = 1$, the **nth roots of unity** $\sqrt[n]{1} = \left(\cos\frac{2kp}{n} + i\sin\frac{2kp}{n}\right)$

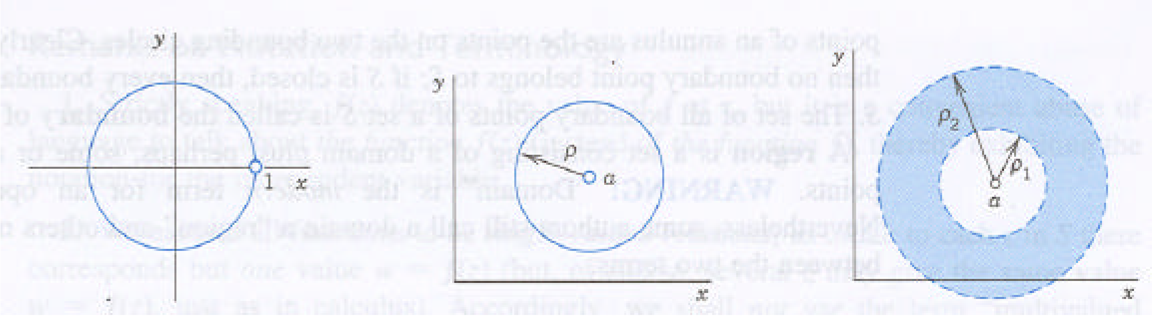
They lie on the **unit circle** of radius 1 and center 0

If w denotes the value corresponding to $k = 1$ then the n values of $\sqrt[n]{1}$ can be written as $1, w, w^2, \cdots, w^{n-1}$

In general, if w_1 is any nth root of an arbitrary complex number $z \neq 0$ then the n values of $\sqrt[n]{z}$ can be written as $w_1, w_1 w, w_1 w^2, \cdots, w_1 w^{n-1}$, because multiplying w_1 by w^k corresponds to increasing the argument of w_1 by $2kp/n$

Circles and disks

Consistent with definition of unit circle, any general circle in complex plane as the form $|z-a| = r$, because it is the set of all z whose distance from the center a equals r



- **Open circular disk (neighborhood** of a**) defined as** $|z-a| < r$
- **Closed circular disk** $|z-a| \le r$
- **Open annulus** $r_1 < |z-a| < r_2$ and **closed annulus** $r_1 \le |z-a| \le r_2$
- **Upper half-plane** $\Rightarrow$ all points $z = x + iy$ with $y > 0$
- **Lower half-plane** $\Rightarrow$ all points $z = x + iy$ with $y < 0$
- **Right half-plane** $\Rightarrow$ all points $z = x + iy$ with $x > 0$
- **Left half-plane** $\Rightarrow$ all points $z = x + iy$ with $x < 0$

Sets in complex plane

- **Points set =** any sort of collection of finitely many or infinitely many points
- **Open set =** every point of S has neighborhood consisting entirely of points of S
- **A connected set** = a set where any two points of the set can be joined by a broken line of finitely many straight-line segments all of whose points belong to the set
  - Open and connected sets are called **domain –** *open disks and open annulus are domains*
- The **complement** of a set S is the set of all points that do not belong to S - a set is closed if its complement is open
- A **boundary point** of a set S is a point every neighborhood of which contains both points that belong to S and points that do not belong to S
  - if the set is open no boundary points belong to the set and if it is closed all boundary points are within the set
- The set of all boundaries of S is called the **boundary** of S
- A **region** is a set consisting of a domain plus perhaps some or all of its boundary points

Complex function

A **complex function** f defined on S is a rule that assigns to every z in S a complex number w called the value of z: $w = f(z)$

The set of all values of a function f is called the **range** of f

If $w = u + iv$ then $w = f(z) = u(x,y) + iv(x,y)$

- *A complex function is equivalent to a pair of real functions*

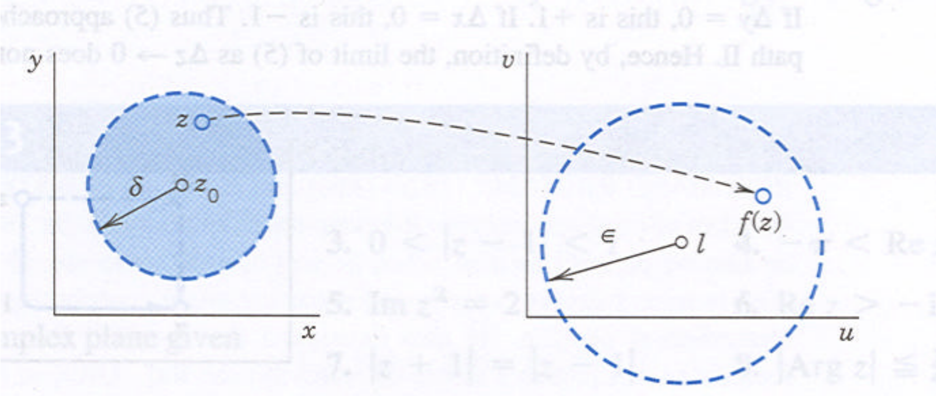
A function $f(z)$ is said to have a **limit** l as z approaches a point z_0 written as $\lim\limits_{z \to z_0} f(z) = l$, if for every positive real $\boldsymbol{e}$ we can find a positive real $\boldsymbol{d}$ such that for all $z \neq z_0$ in the disk $|z - z_0| < \boldsymbol{d}$ we have $|f(z) - l| < \boldsymbol{e}$

- ***Definition similar to real function with difference that in complex plane*** z ***may approach*** z_0 ***from any direction***

- ***If the limit exists it is unique***

A function $f(z)$ is continuous at $z = z_0$ if $f(z_0)$ is defined and $\lim\limits_{z \to z_0} f(z) = f(z_0)$

$f(z)$ is continuous on a domain if it is continuous at each point of the domain



The **derivative of a complex function** $f(z)$ at a point z_0 is written $f'(z_0)$ and is defined as $f'(z_0) = \lim_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

Provided the limit exists, then $f(z)$ is said to be **differentiable** at z_0

Writing $\Delta z = z - z_0 \Rightarrow z = z_0 + \Delta z$ then we have the alternative definition

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

- Differentiability at z_0 means that along whatever path z approaches z_0 the quotient $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ approaches a certain value and all these values are equals

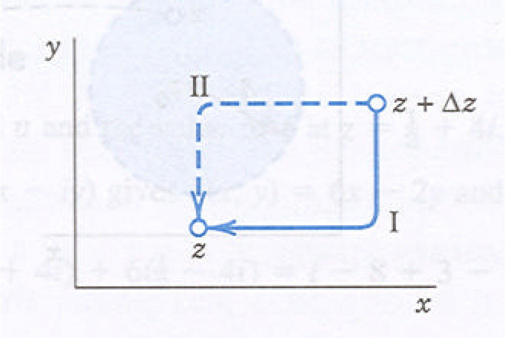
Differential rules:

- $(cf)' = cf'$
- $(f+g)' = f' + g'$
- $(fg)' = f'g + fg'$
- $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$
- Also chain rule and power rule $\left(z^n\right)' = nz^{n-1}$ with n an integer
- If $f(z)$ is differentiable at z_0 it is continuous at z_0

IMPORTANT: there are many complex functions that are not differentiable

For instance $f(z) = \overline{z}$

Write $\Delta z = \Delta x + i\Delta y$ then $\dfrac{f(z+\Delta z) - f(z)}{\Delta z} = \dfrac{\overline{(z+\Delta z)} - \overline{z}}{\Delta z} = \dfrac{\overline{\Delta z}}{\Delta z} = \dfrac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$



If $\Delta y = 0$ this is $+1$ along path I and if $\Delta x = 0$ this is -1 along path II
Therefore $\Delta z \to 0$ does not exists at any z

- ***Differentiability of a complex function is rather a severe requirement***

A function $f(z)$ is said to be **analytic** in a domain D (or **holomorphic in** D) if $f(z)$ is defined and differentiable at all points of D

A function $f(z)$ is said to be **analytic** at a point z_0 in D if $f(z)$ is analytic in a neighborhood of z_0

Cauchy-Riemann and Laplace's equations

The Cauchy-Riemann equations = pillars of complex analysis – provide criterion for analyticity of complex functions $w = f(z) = u(x,y) + iv(x,y)$

The function $f(z)$ is **analytic** in a domain D if and only if the first partial derivative of $u(x,y)$ and $v(x,y)$ satisfy the **Cauchy-Riemann equations**: $u_x = v_y$ and $u_y = -v_x$

Where $u_x = \dfrac{\partial u}{\partial x}$ and $u_y = \dfrac{\partial u}{\partial y}$ are the partial derivative of the function

Example: $f(z) = z^2 = x^2 - y^2 + 2ixy$
Here $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$
Therefore $u_x = 2x = v_y$ and $u_y = -2y = -v_x$

Theorem 1 Cauchy Riemann equations

Let $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself

Then at that point the first-order partial derivatives of $u(x,y)$ and $v(x,y)$ exist and satisfy the Cauchy-Riemann equations

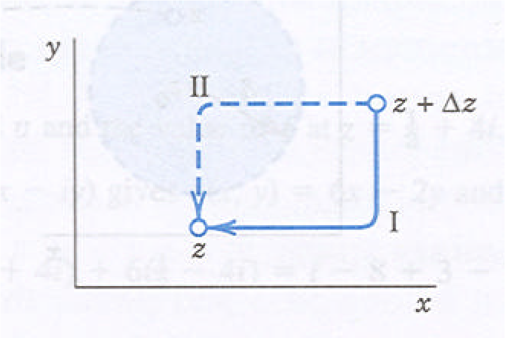
Hence, if $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D those partial derivatives exist and satisfy the Cauchy-Riemann equations at all points of D

PROOF

The assumption is that $f'(z)$ exists

Then by definition $f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

Let Δz approaches zero along any path, example path I and path II, in the neighborhood of z - and by definition the results must be the same



Since $\Delta z = \Delta x + i\,\Delta y \Rightarrow z + \Delta z = x + \Delta x + i\,(y + \Delta y)$ and the

derivative $f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{\left[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\right] - \left[u(x,y) + iv(x,y)\right]}{\Delta x + i\Delta y}$

Choosing path I: $\Delta y \to 0$ first and then $\Delta x \to 0$; putting $\Delta y = 0$ and $\Delta z = \Delta x$

thus $f'(z) = \lim\limits_{\Delta x \to 0} \dfrac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim\limits_{\Delta x \to 0} \dfrac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$

Because by assumption the two limits exist then we have $f'(z) = u_x + iv_x$

Choosing path II $\Delta x \to 0$ first and then $\Delta y \to 0$; putting $\Delta x = 0$ and $\Delta z = i\Delta y$

thus $f'(z) = \lim\limits_{\Delta y \to 0} \dfrac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim\limits_{\Delta y \to 0} \dfrac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$

Because by assumption the two limits exist then we have $f'(z) = -iu_y + v_y$

This shows that the existence of the derivative $f'(z)$ implies the existence of 4 partial derivatives – equating the real and complex part we have the conditions:

Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$

Theorem 2

If 2 real-valued continuous functions $u(x,y)$ and $v(x,y)$ of 2 real variables x and y have continuous first partial derivatives that satisfy the Cauchy-Riemann equations in some domain D then the complex function $f(z) = u(x,y) + iv(x,y)$ is analytic

- ***Consequence = the Cauchy-Riemann equations not only necessary conditions but also sufficient***

Example: $f(z) = \overline{z} = x - iy$

Here $u = x$ and $v = y \Rightarrow u_y = -v_x$ but $u_x = 1 \neq v_y = -1$, the function is not analytic

Example: $f(z) = e^x(\cos y + i\sin y)$

Here $u = e^x \cos y$ and $v = e^x \sin y$ such that $u_x = e^x \cos y = v_y$ and

$u_y = -e^x \sin y = -v_x = e^x \sin y$ therefore $f(z) = e^x(\cos y + i\sin y)$ is analytic

Polar form

$z = r(\cos\theta + i\sin\theta)$ where here $u = r\cos\theta$ and $v = r\sin\theta$

Applying the chain rule $u_r = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} = \cos\theta$ and $u_\theta = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \theta} = -r\sin\theta$

Similarly $v_r = \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial r} = \sin\theta$ and $v_\theta = \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial \theta} = r\cos\theta$

such that $u_r = \dfrac{1}{r}v_\theta$ and $v_r = -\dfrac{1}{r}u_\theta$

Importance of complex functions

- ***Both Im and Re parts of analytic function satisfy Laplace's equation***

Laplace's equation

If $f\left(z\right)=u\left(x,y\right)+iv\left(x,y\right)$ is analytic in a domain D then both u and v satisfy Laplace's equation $\nabla^2 u=u_{xx}+u_{yy}=0$ and $\nabla^2 v=v_{xx}+v_{yy}=0$ in D and have continuous second partial derivatives in D

PROOF
According to Cauchy-Riemann equations $u_x=v_y$ and $u_y=-v_x$
Taking the second derivative $u_{xx}=v_{yx}$ and $u_{yy}=-v_{xy}$

By definition the derivative of an analytic function is itself analytic, which implies that u and v have continuous partial derivative of all orders and $v_{xy}=v_{yx}$

Summing together $u_{xx}=v_{yx}$ with $u_{yy}=-v_{xy}$ we thus get $\nabla^2 u=u_{xx}+u_{yy}=v_{yx}-v_{xy}=0$

Similarly $u_{xy}=v_{yy}$ and $u_{yx}=-v_{xx}$ such that $\nabla^2 v=v_{xx}+v_{yy}=-u_{yx}+u_{xy}=0$

Solutions of Laplace's equations having continuous second order partial derivatives are called **harmonic functions** and this theory is called **potential theory**

- ***The real and imaginary part of analytic functions are harmonic functions***

If 2 harmonic functions u and v satisfy the Cauchy-Riemann equations in a domain D they are real and imaginary parts of an analytic function f in D

v is the **harmonic conjugate function** of u in D

Example: $u = x^2 - y^2 - y$

The real part satisfies the Laplace's equation $u_x = 2x \Rightarrow u_{xx} = 2$ and $u_y = -2y - 1 \Rightarrow u_{yy} = -2$ such that $\nabla^2 u = 0$

We search the conjugate – by definition $v_y = u_x = 2x$ and $v_x = -u_y = 2y + 1$
Integrating the first equation $v = 2xy + h(x)$ then differentiating

$$v_x = 2y + \frac{dh(x)}{dx} \Rightarrow \frac{dh(x)}{dx} = 1 \Rightarrow h(x) = x + c \text{ where } c \in \mathbb{R}$$

The harmonic conjugate $v = 2xy + x + c$ and the harmonic function

$$f(z) = x^2 - y^2 - y + i(2xy + x + c) = z^2 + iz + ic$$

- ***A conjugate of a given harmonic function is uniquely determined up to an arbitrarily real additive constant***

Basic elementary complex functions

Counter parts to functions in calculus, which are reduce to $z = x \in \mathbb{R}$

Exponential function $e^z = \exp z$

$$e^z = e^x (\cos y + i \sin y)$$

Properties:

- $y = 0 \Rightarrow e^z = e^x$
- e^z analytic for all z ➔ **Entire function** – a function which is analytic for all z
- $(e^z)' = e^z$
- $e^{z_1 + z_2} = e^{z_1} e^{z_2}$

Special case $z_1 = x$ and $z_2 = iy$ then $e^z = e^x e^{iy}$

- **Euler Formula**: $e^{iy} = \cos y + i \sin y$
- In polar form $z = re^{iq}$ and putting $y = 2p \Rightarrow e^{2pi} = 1$

Other formulas: $e^{\frac{p}{2}i} = i, e^{-\frac{p}{2}i} = -i, e^{pi} = e^{-pi} - 1$

Another important consequence of Euler formula
$\left|e^{iy}\right| = \left|\cos y + i \sin y\right| = \sqrt{\cos^2 y + \sin^2 y} = 1$

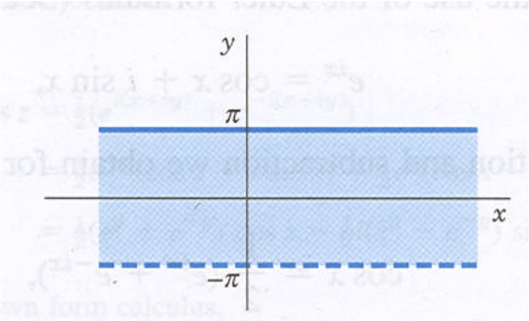
- pure imaginary exponential have absolute value 1

Therefore $\left|e^z\right| = e^x$ and $\arg e^z = y \pm n2p$ where $n = 0,1,2,...$

- Since $\left|e^z\right| = e^x \neq 0 \Rightarrow e^z \neq 0$, this is a function that never vanish

- $\Rightarrow e^{z+2pi} = e^z$ for all z; all the values that $w = e^z$ can assume are already assumed in the horizontal strip of width $2p$ = **fundamental region**

Fundamental region of e^z, $-p < y \leq p$



Example: To solve $e^z = 3 + 4i$, first we note that $\left|e^z\right| = e^x = 5 \Rightarrow x = \ln 5 = 1.609$

Since $e^x \cos y = 3 \Rightarrow \cos y = 0.6$ and $e^x \sin y = 4 \Rightarrow \sin y = 0.8$, thus $y = 0.927$
And $z = 1.609 + 0.927i \pm 2npi$ with $n = 0,1,2,...$

We see that many properties of e^z parallel e^x; an exception is the periodicity of e^z with $2pi$, which suggested the concept of fundamental region

Trigonometric and hyperbolic functions

Using Euler formula: $e^{ix} = \cos x + i\sin x$ and $e^{-ix} = \cos x - i\sin x$

Adding together $\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$

Subtracting $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$

Therefore for complex values $z = x + iy$

$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$ and $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

Since e^z is entire, so are $\cos z$ and $\sin z$

NOTE for real number we had to invent a new type of function – **hyperbolic functions**

$\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$

This is not the case in complex: ***in complex, functions come together that are unrelated in real – this is typical of general situation and one advantage of working in complex***

Other trigonometric complex functions:

$\tan z = \frac{\sin z}{\cos z}$, $\cot z = \frac{\cos z}{\sin z}$, $\sec z = \frac{1}{\cos z}$ and $\csc z = \frac{1}{\sin z}$

- Analytic except where $\cos z = 0$ and $\sin z = 0$

Taking the derivative $(\cos z)' = \frac{1}{2}\left(ie^{iz} - ie^{iz}\right) = -\frac{1}{2i}\left(e^{iz} - e^{iz}\right) = -\sin z$

- Euler formula also valid $e^{iz} = \cos z + i\sin z$

The general formulae for trigonometric functions also hold:

- $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$
- $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$
- But also $\cos^2 z + \sin^2 z = 1$

Hyperbolic complex functions

Same as in real: $\cosh z = \frac{1}{2}\left(e^{z} + e^{-z}\right)$ and $\sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right)$

Entire functions with derivatives
$\cosh' z = \sinh z$ and $\sinh' z = \cosh z$

As usual we also have
$\tanh z = \frac{\sinh z}{\cosh z}$, $\coth z = \frac{\cosh z}{\sinh z}$, $\operatorname{sech} z = \frac{1}{\cosh z}$ and $\operatorname{csch} z = \frac{1}{\sinh z}$

- ***Contrary than in real, the trigonometric and hyperbolic functions are simply related*** we have $\cosh iz = \cos z$ and $\sinh iz = i \sin z$ or alternatively
  $\cos iz = \cosh z$ and $\sin iz = i \sinh z$

- ***Advantage of complex: gives a uniform + deeper understanding of special functions***

Logarithm

$\ln z$ is the **natural logarithm** of $z = x + iy$

Therefore $w = \ln z \Rightarrow e^{w} = z$

By definition $z = 0$ is impossible because $e^{w} \neq 0$ for all w

If we set $w = u + iv$ and $z = re^{iq}$ this becomes $e^{w} = e^{u+iv} = re^{iq}$
And since $\left|e^{w}\right| = e^{u} \Rightarrow r = e^{u}$ and the argument is $v = q$

Since $e^{u} = r \Rightarrow u = \ln r$, the natural logarithm of positive number

Hence $w = u + iv = \ln z \Rightarrow \ln z = \ln r + iq$, where $|r| > 0$ and $q = \arg z$, which is determined up to integer multiple of $2p$

Therefore the complex natural logarithm $\ln z$ with $z \neq 0$ ***is infinitely many-valued***

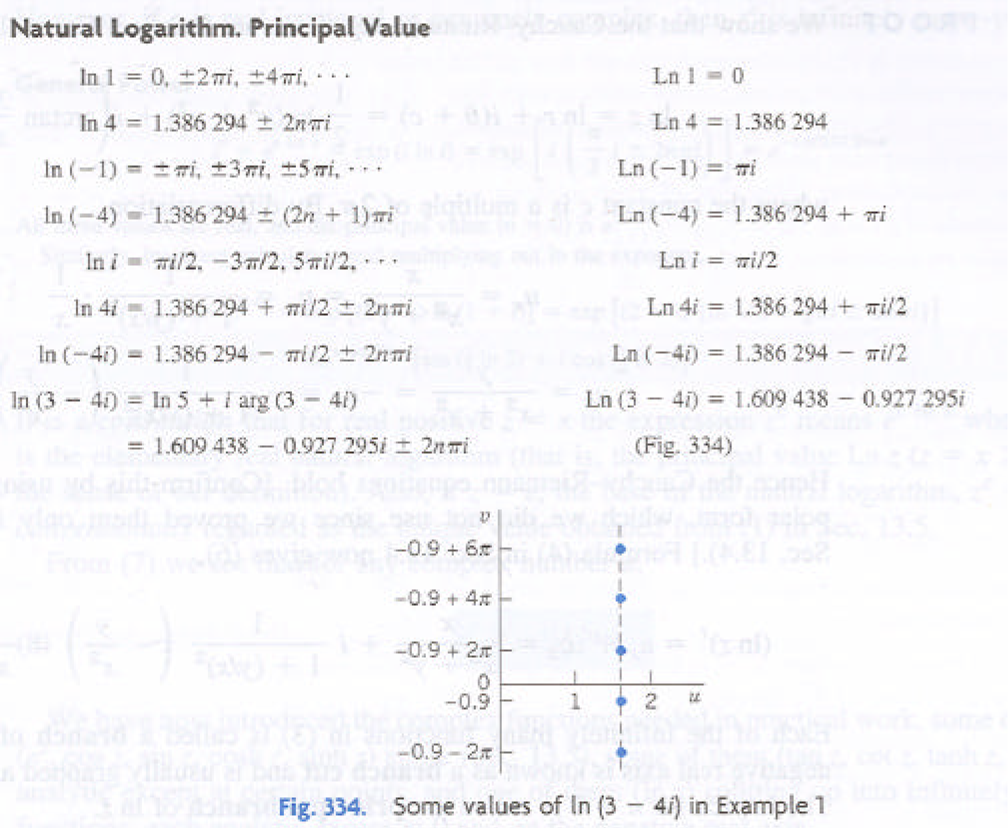
Principal value $\text{Ln}\, z = \ln|z| + i\,\text{Arg}\, z \;\Rightarrow \ln z = \text{Ln}\, z \pm 2np\, i$ with $n = 1,2,...$

- They all have the same real part, but the imaginary part differ by multiples of $2p$

If $z > 0$ and real, then $\text{Arg}\, z = 0$ and $\text{Ln}\, z$ is the real natural logarithm
If $z < 0$ and real (real natural logarithm not defined) then $\text{Arg}\, z = p$ and $\text{Ln}\, z = \ln|z| + p\, i$

Since $e^{\ln r} = r$ for r positive and real we obtain $e^{\ln z} = z$ which is a multi-valued since $\arg\left(e^{z}\right) = y \pm 2np$

That is $\ln\left(e^{z}\right) = z \pm 2np\, i$, with $n = 0,1,2,...$



Natural Logarithm. Principal Value

$\ln 1 = 0, \pm 2\pi i, \pm 4\pi i, \cdots$	$\text{Ln}\, 1 = 0$
$\ln 4 = 1.386\,294 \pm 2n\pi i$	$\text{Ln}\, 4 = 1.386\,294$
$\ln(-1) = \pm\pi i, \pm 3\pi i, \pm 5\pi i, \cdots$	$\text{Ln}\,(-1) = \pi i$
$\ln(-4) = 1.386\,294 \pm (2n+1)\pi i$	$\text{Ln}\,(-4) = 1.386\,294 + \pi i$
$\ln i = \pi i/2, -3\pi/2, 5\pi i/2, \cdots$	$\text{Ln}\, i = \pi i/2$
$\ln 4i = 1.386\,294 + \pi i/2 \pm 2n\pi i$	$\text{Ln}\, 4i = 1.386\,294 + \pi i/2$
$\ln(-4i) = 1.386\,294 - \pi i/2 \pm 2n\pi i$	$\text{Ln}\,(-4i) = 1.386\,294 - \pi i/2$
$\ln(3-4i) = \ln 5 + i \arg(3-4i)$	$\text{Ln}\,(3-4i) = 1.609\,438 - 0.927\,295i$
$= 1.609\,438 - 0.927\,295i \pm 2n\pi i$	(Fig. 334)

Fig. 334. Some values of ln (3 − 4i) in Example 1

Again, the familiar relations continue to hold

$$\ln(z_1z_2) = \ln z_1 + \ln z_2 \quad \text{and} \quad \ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$$

But it is understood that each value of one side is also contained among the values of the other side

Example: $z_1 = z_2 = e^{\pi i} = -1$, taking the principal value $\operatorname{Ln} z_1 = \operatorname{Ln} z_2 = \pi i$, therefore $\ln(z_1z_2) = \ln 1 = 2\pi i$, but it is not true for $\operatorname{Ln}(z_1z_2) = \operatorname{Ln} 1 = 0$

Theorem: Analyticity of logarithm

For every $n = 0, \pm 1, \pm 2, \ldots$ the function $\ln z = \operatorname{Ln} z \pm 2n\pi i$ is analytic (except at zero and on the negative real) axis and has the derivative $\ln' z = \dfrac{1}{z}$

PROOF – the Cauchy-Riemann equations must be satisfied

$$\ln z = \ln r + i(\theta + c) = \frac{1}{2}\ln\left(x^2 + y^2\right) + i\left(\arctan\frac{y}{x} + c\right)$$

Therefore $u_x = \dfrac{x}{x^2+y^2} = v_y = \dfrac{1}{1+\left(\dfrac{y}{x}\right)^2} \cdot \dfrac{1}{x}$

And $u_y = \dfrac{y}{x^2+y^2} = -v_x = -\dfrac{1}{1+\left(\dfrac{y}{x}\right)^2}\left(-\dfrac{y}{x^2}\right)$

Thus $\ln' z = u_x + iv_x = \dfrac{x}{x^2+y^2} + i\left(-\dfrac{y}{x^2+y^2}\right) = \dfrac{x-iy}{x^2+y^2} = \dfrac{1}{z}$

Each of the infinitely many functions is called a **branch** of the logarithm

The negative real axis is known as a **branch cut**

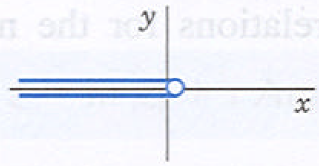


Fig. 335. Branch cut for ln z

The branch for $n = 0$ is the **principal branch** of $\ln z$

General Powers

For c complex and $z = x + iy \neq 0$, **general power** $z^c = e^{c\ln z}$

Since $\ln z$ is infinitely many-valued $\Rightarrow$ z^c will in general be multi-valued

Principal value of z^c is $z^c = e^{c\mathrm{Ln}z}$

- If $c = n = 1,2,3,\ldots$ then z^n is single-valued and identical with the usual nth power of z
- If $c = -1,-2,\ldots$ the situation is similar
- If $c = \dfrac{1}{n}$ where $n = 2,3,\ldots$ then $z^c = \sqrt[n]{z} = e^{\left(\frac{1}{n}\right)\ln z}$ the exponent is determined up to multiples of $2pi/n$
- If $c = \dfrac{p}{q}$ the quotient of two positive integers, the situation is similar and z^c has only finitely many distinct values
- If c is real irrational or genuinely complex then z^c is infinitely many-valued

Examples

$$i^i = e^{i\ln i} = \exp(i\ln i) = \exp\left[i(i\frac{p}{2} \pm 2np\, i)\right] = e^{-\left(\frac{p}{2}\pm 2np\right)}$$

$$\left(1+i\right)^{2-i} = \exp\left[\left(2-i\right)\ln\left(1+i\right)\right] = \exp\left[\left(2-i\right)\left(\ln\sqrt{2}+\frac{1}{4}p\, i \pm 2np\, i\right)\right] =$$
$$= 2e^{p/4\pm 2np}\left[\sin\left(\frac{1}{2}\ln 2\right)+i\cos\left(\frac{1}{2}\ln 2\right)\right]$$

Convention: for real positive $z = x$, $z^c = e^{c\ln x}$ and if $z = e \Rightarrow z^c = e^c$ is a unique value

For any complex number a, $a^z = e^{z\ln a}$

Summary of complex functions:

$e^z, \cos z, \sin z, \cosh z, \sinh z$	Entire functions
$\tan z, \cot z, \tanh z, \coth z$	Analytic except at certain points
$\ln z$	Infinitely many functions each analytic except at 0 and on negative real axis