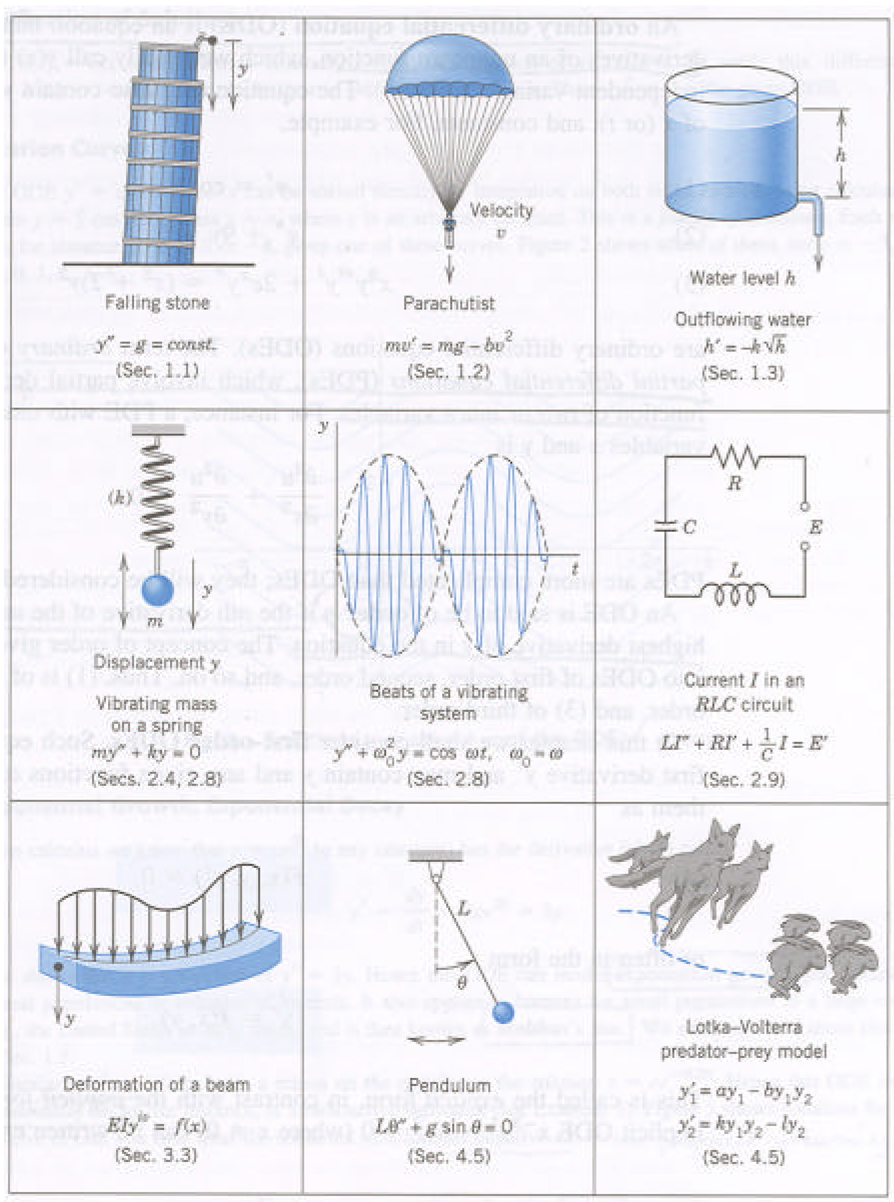
Differential equations

First order differential equation

Implicit $F(x,y,y')=0$ or **Explicit** $y'=f(x,y)$

In general = physical laws involving a rate of change (with time) of an event = function



Solution of DE = Function $y = h(x)$

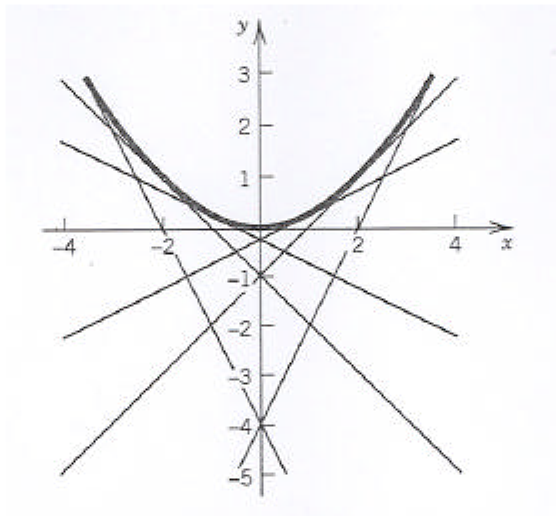
- Derivable ➔ $y' = h'(x)$ and satisfy DE on interval $a < x < b$
- Conditions for DE to have a solution are general
- Many simple DE do not have a solution Ex. $(y')^2 = -1$ ➔ no solution $\in \mathbb{R}$

Ex. 1

$$(y')^2 - xy' + y = 0$$

General solution:
$y = cx - c^2$ ➔ *family of straight lines;*

Singular solution: $y = \dfrac{x^2}{4}$ ➔ *parabola*



General solution: many solutions differing by constant ➔ **family of curves**
Singular solution: 1 additional solution not obtained from general solution

Ex. 2 Radioactive decay

Observed ➔ *decay with time of radioactive substance* $\propto$ *to amount present*

➔ $\dfrac{dy}{dt} = ky$

k is a constant (half life) depending on radioactive substance; since $\dfrac{dy}{dt} < 1$ ➔ $k < 0$

General solution: $y = ce^{kt}$

Verification: $y' = cke^{kt} = ky(t)$

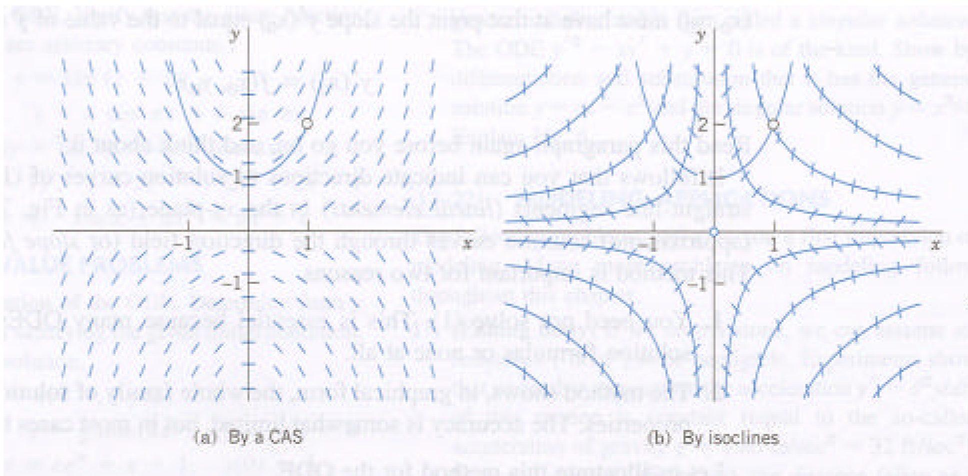
Particular solution: at $t = 0$, $y(0) = Y_0 \Rightarrow y(0) = ce^0 = c \Rightarrow c = Y_0 \Rightarrow y(t) = Y_0 e^{kt}$

Geometrical interpretations

$y' = f(x,y)$ is the slope of $y(x)$ ➔ if $y(x)$ solution passing through (x_0, y_0) of xy-plane then slope at that point is $f(x_0, y_0)$ ➔ $f(x,y)$ = **direction field** (or slope field) of solution curves

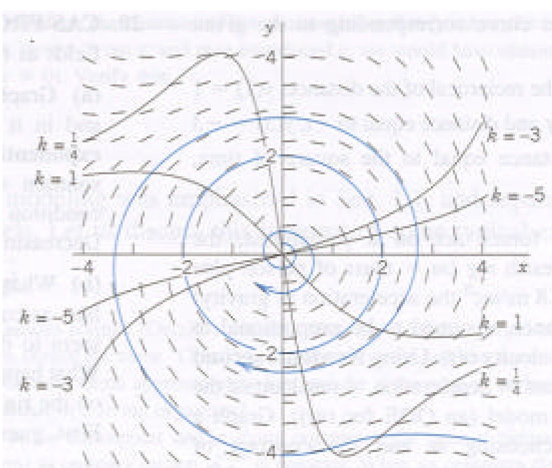
Isoclines: $y' = f(x,y) = k =$ constant

Ex. 1 $y'(x) = xy$, the isoclines are hyperbolas


(a) By a CAS (b) By isoclines

General solution exist: $y = ce^{x^2/2}$; verification: $y' = x(ce^{x^2/2}) = xy$

Ex. 2 van der Pol equation of electronics $y' = 0.1(1 - x^2) - \dfrac{x}{y}$



Direction fields give all solutions with limited accuracy

More accurate method = numeric method
Allows to solve DE for which we have no solution

Ex **Euler-Cauchy method** and **Runge-Kutta method**

Methods to find general solutions

Case 1: Separable DE

$g(y)y' = f(x) \Rightarrow g(y)dy = f(x)dx$

Solution: $\int g(y)\frac{dy}{dx}dx = \int f(x)dx + C$ but since $\frac{dy}{dx}dx = dy \Rightarrow \int g(y)dy = \int f(x)dx + C$

Integrals exist provided g and f continuous

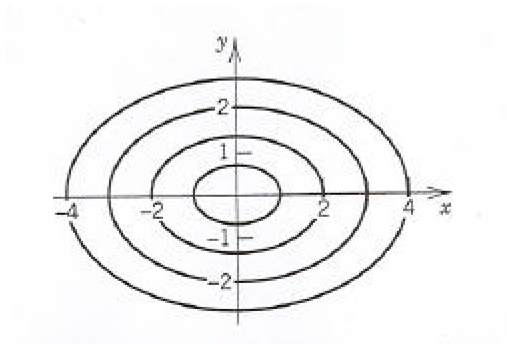
Ex. 1

$9yy' + 4x = 0 \Rightarrow 9ydy + 4xdx = 0$

Solution:

$\int 9ydy + \int 4xdx + C_1 = 0$

or $\frac{y^2}{4} + \frac{x^2}{9} = C_2$



family of ellipses centered on (0,0)

Ex. 2

$y' = 1 + y^2 \Rightarrow \int \frac{dy}{1+y^2} = \int dx + C$

Solution: $\arctan y = x + C \Rightarrow y = \tan(x + C)$

Note $y = \tan x + C$ is not a solution

Ex. 3

$y' = ky$ with $|k| > 0$

Solution: $\int \frac{dy}{y} = k\int dx + C_1 \Rightarrow \ln|y| = kx + C_1 \Rightarrow |y| = e^{kx+C_1} = Ce^{kx}$

Where $C = +e^{C_1}$ for $y > 0$ and $C = -e^{C_1}$ $y < 0$

Ex. 4 Initial value problem – initial condition (IC)

DE: $y' = \frac{-y}{x}$ IC: $y(1) = 1$

Solution: $\frac{dy}{y} = -\frac{dx}{x} \Rightarrow \ln|y| = -\ln|x| + C_1 = \ln\frac{1}{|x|} + C_1 \Rightarrow y = \frac{C}{x}$

IC ➔ $1 = \frac{C}{1} \Rightarrow C = 1 \Rightarrow y = \frac{1}{x}$

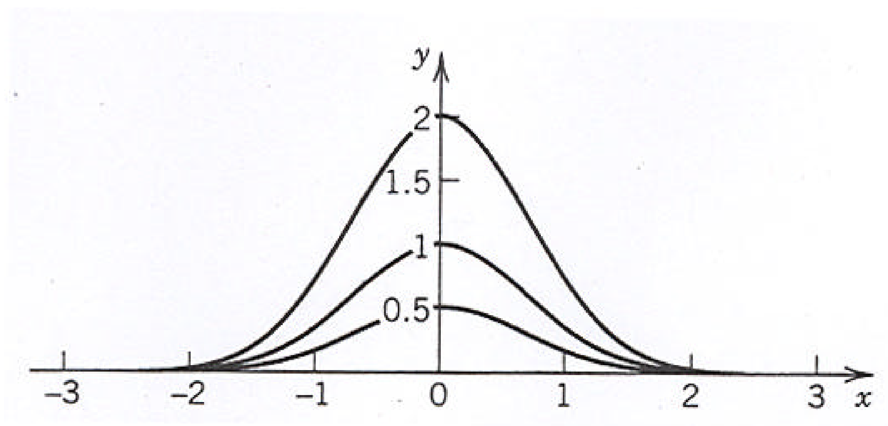
Ex. 5 Initial value problem

DE: $y' = -2xy$ IC: $y(0) = 1$

Solution: $\frac{dy}{y} = -2xdx \Rightarrow \ln|y| = -x^2 + C_1 \Rightarrow |y| = e^{-x^2+C_1} = Ce^{-x^2}$

Where $\begin{cases} e^{C_1} = +C \text{ for } y > 0 \\ e^{C_1} = -C \text{ for } y < 0 \\ C = 0 \text{ for } y = 0 \end{cases}$

IC: $y(0) = 1 \Rightarrow y = e^{-x^2}$ ➔ **Bell-shaped curves**



Case 2: Reduction to separable form

$$y' = g\left(\frac{y}{x}\right)$$

Solution ➔ use transformation $u = \frac{y}{x} \Rightarrow y = ux$

$$\Rightarrow y' = u'x + u \Rightarrow u'x + u = g(u) \Rightarrow u'x = g(u) - u$$

$$\Rightarrow \frac{du}{g(u) - u} = \frac{dx}{x}$$

Ex. 1

$$2xyy' = y^2 - x^2 \Rightarrow y' = \frac{y}{2x} - \frac{x}{2y} = \frac{1}{2}\left(\frac{y}{x} - \frac{x}{y}\right)$$

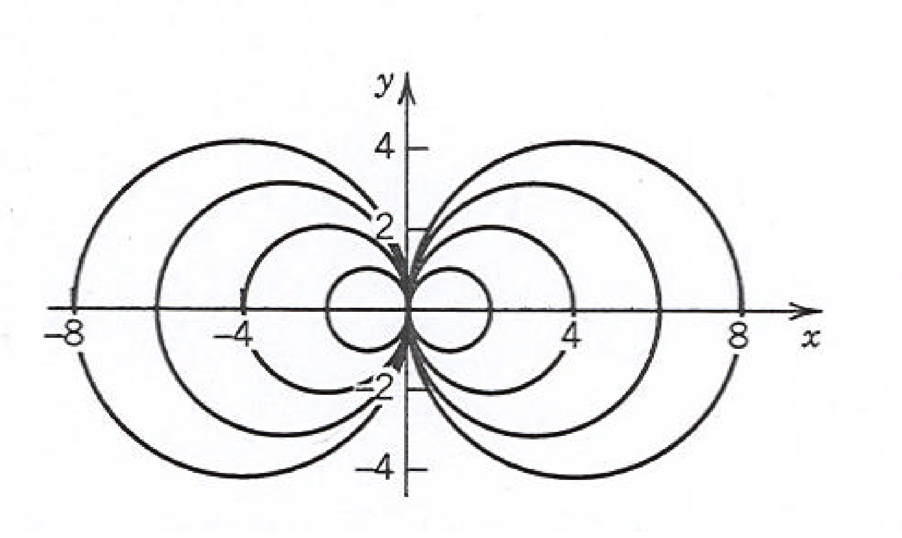
$$\Rightarrow u'x + u = \frac{1}{2}\left(u - \frac{1}{u}\right) \Rightarrow u'x = -\frac{1}{2}\left(u + \frac{1}{u}\right) = -\left(\frac{u^2 + 1}{2u}\right)$$

$$\Rightarrow \frac{2udu}{1+u^2} = -\frac{dx}{x} \Rightarrow \ln\left|1 + u^2\right| = -\ln|x| + C_1$$

$$\Rightarrow 1 + u^2 = \frac{C}{x} \Rightarrow 1 + \frac{y^2}{x^2} = \frac{C}{x} \Rightarrow x^2 + y^2 = xC$$

$$\Rightarrow \left(x - \frac{C}{2}\right)^2 + y^2 = \frac{C^2}{4}$$

Family of circles, tangent to origin, with centers on x-axis



Construction of mathematical models

Radio carbon dating

Based on isotope ${}_6C^{14}$

- Rendered radioactive by cosmic rays bombardment
- In the air the ratio of the isotopes C^{14}/C^{12} is constant

Principle of method:
When living organism dies the absorption (breathing and eating) of ${}_6C^{14}$ ends
➔ The ratio of C^{14}/C^{12} measured compared to value in air $\propto$ to date of death

Half life of ${}_6C^{14}$ is 5730 yrs ± 40 yrs (error of 7%)

Mathematical model:
$y' = ky \Rightarrow y(t) = y_0 e^{kt}$ where y_0 is the initial amount of ${}_6C^{14}$

From definition of half life ➔ time at which the level is ½ the original one:

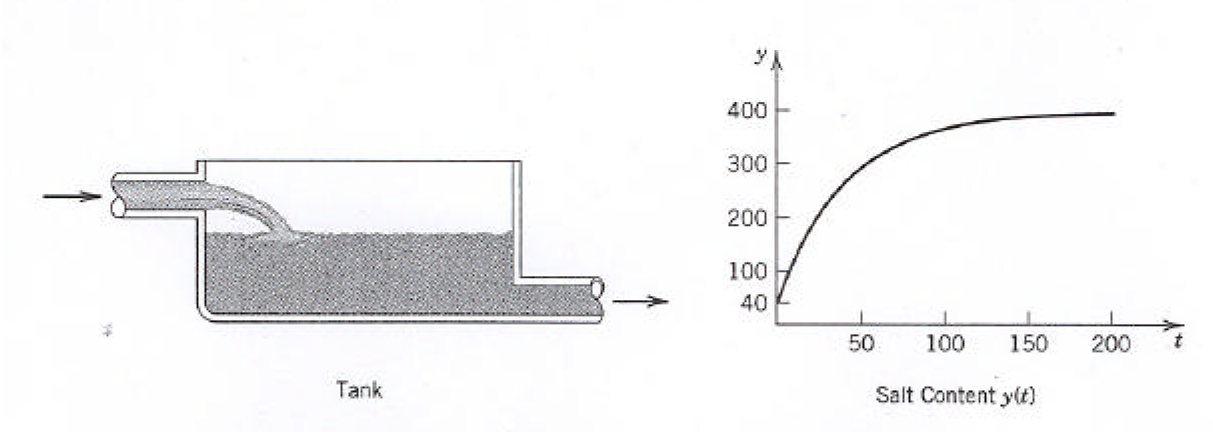
$$\Rightarrow y_0 e^{k\cdot 5730} = \frac{1}{2} y_0 \Rightarrow e^{k\cdot 5730} = \frac{1}{2} \Rightarrow k = \frac{1}{5730} \ln\frac{1}{2} \approx -0.000121 \text{ yr}^{-1}$$

Measurement: if remains contain 25% of original level

$$\Rightarrow y_0 e^{-0.000121t} = 0.25 y_0 \Rightarrow t = \frac{\ln 0.25}{-0.000121} \text{ yrs} \approx 11460 \pm 80 \text{yrs}$$

Comparisons with other dating methods suggest $t \sim 12000 - 13000$ yrs
➔ value given by model is too low, why?

Mixing problem



A thank contains 200 gal of water in which 40 lb of salt is dissolved. Each minute 5gal of water, with 2lb/gal of salt is added, and the same amount is retrieved

Problem: amount of salt as a function of time: $y(t)$ lb

Simplification: the mixing process is instantaneous and perfect

Mathematical model: rate of change $y'(t) = \frac{dy}{dt} = \text{inflow} - \text{outflow}$

Inflow: 10 lb/min

Outflow: since one gal of water contains $\frac{y(t)}{200} \frac{\text{lb}}{\text{gal}}$

5 gal coming out will contain $\frac{5y(t)}{200} = \frac{y(t)}{40} = 0.025y(t)\frac{\text{lb}}{\text{min}}$

Rate of change: $y' = 10 - 0.025y(t)$

IC: $y(0) = 40$ lb

Solution:

$$y' = -0.025(y-400) \Rightarrow \frac{dy}{(y-400)} = -0.025dt$$

$$\Rightarrow \ln|y-400| = -0.025t + C_1 \Rightarrow y = Ce^{-0.025t} + 400$$

IC: $$y(0) = 40 \Rightarrow C = -360$$
$$\Rightarrow y(t) = 400 - 360e^{-0.025t}$$

Solution increases with time from 40 to 400.
Equilibrium reached when amount going in = amount going out ➔ $y' = 0$;

Heating problem (Newton's law of cooling)

The time rate of change of temperature of a body is described by the following equation:

$$\frac{dT}{dt} \propto \Delta T = T_{body} - T_{Medium}$$

Model assuming ambient temperature constant at $T_{medium} = 32^\circ \text{F}$

$$\frac{dT}{dt} = k\left(T_{body} - 32\right) \Rightarrow \frac{dT}{\left(T_{body} - 32\right)} = kdt$$

$$\Rightarrow \ln\left|T_{body} - 32\right| = kt + C_1 \Rightarrow T_{body} = 32 + Ce^{kt}$$

IC:

$$T(0) = T_0 = 66^\circ \text{F} \Rightarrow 32^\circ \text{F} + C = 66^\circ \text{F} \Rightarrow C = 34^\circ \text{F}$$

$$\Rightarrow T\left(t\right) = 32^\circ \text{F} + \left(34^\circ \text{F}\right)e^{kt}$$

Observation: after $t = 2\,\text{h}$ $T_{body} = 63^\circ \text{F}$

$$\Rightarrow T\left(2\text{h}\right) = 32^\circ \text{F} + 34^\circ \text{F}e^{2k} = 63^\circ \text{F} \Rightarrow e^{2k} = \frac{63-32}{34} \approx 0.911765$$

$$\Rightarrow k\ \left(\text{h}^{-1}\right) = \frac{1}{2}\ln 0.911765 = -0.046187\ \text{h}^{-1}$$

Rate of change of temperature decreases with time as the body temperature approach temperature of ambient ➔ Equilibrium = temperature with the ambient

For $t = 10\,\text{h}$ $T\left(10\ \text{h}\right) = 32^\circ \text{F} + 34^\circ \text{F}e^{-0.046187 \cdot 10} = 53.4^\circ \text{F}$

Velocity escape from Earth

MODEL = Newton's law of gravitation $ma = -GMm/r^2$

At the surface of Earth with radius R: $a = g = GM/R^2$

At any radius $a(r) = -\dfrac{gR^2}{r^2}$

Applying definition of acceleration and chain rule $\Rightarrow a(r) = \dfrac{dv}{dt} = \dfrac{dv}{dr}\dfrac{dr}{dt} = \dfrac{dv}{dr}v = -g\dfrac{R^2}{r^2}$

Separation of variable DE $\Rightarrow vdv = -gR^2\dfrac{dr}{r^2} \Rightarrow \dfrac{v^2}{2} = g\dfrac{R^2}{r} + C$

IC: on surface of the Earth $r = R$ and $v = v_0 \Rightarrow C = \dfrac{v_0^2}{2} - gR$

General solution: $v^2 = \dfrac{2gR^2}{r} + v_0^2 - 2gR$

When $v = 0$ the projectile stops and reverse trajectory (fall back to Earth)

However, if we take $v_0 = \sqrt{2gR}$, then $\dfrac{v_0^2}{2} - gR = 0$ and $v^2 = \dfrac{2gR^2}{r} > 0$ and v remains positive reaching 0 only at the infinite

Escape velocity: $v_0 = \sqrt{2gR}$

Exact DE

General form: $M(x, y)dx + N(x, y)dy = 0$

Solution: $u(x, y) = C$, such that $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = Mdx + Ndy = 0$

Condition (necessary and sufficient): M and N have continuous first partial derivatives and $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial M}{\partial y}$

Solution: if $M(x, y)dx + N(x, y)dy = 0$ then $u = \int Mdx + k(y)$ where $k(y)$ plays the role of constant ➔ to determine $k(y)$ derive $\frac{\partial u}{\partial y}$ to get $\frac{dk}{dy}$ and integrate.

Note: we could have started with $u = \int Ndy + l(x)$

Ex. 1 $\left(x^3 + 3xy^2\right)dx + \left(3x^2 y + y^3\right)dy = 0$

Verification: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 6xy$

First integration: $u = \int \left(x^3 + 3xy^2\right)dx + k(y) \Rightarrow \frac{1}{4}x^4 + \frac{3}{2}x^2 y^2 + k(y)$

Determination of k: $\frac{\partial u}{\partial y} = 3x^2 y + \frac{dk}{dy} = 3x^2 y + y^3 \Rightarrow \frac{dk}{dy} = y^3 \Rightarrow k = \frac{y^4}{4} + C_1$

General solution: $\frac{1}{4}x^4 + \frac{3}{2}x^2 y^2 + \frac{1}{4}y^4 = C$

Ex. 2 $(\sin x \cosh y)dx - (\cos x \sinh y)dy = 0$, with IC $y(0) = 3$

Verification: $\dfrac{\partial M}{\partial y} = \sin x \sinh y = \dfrac{\partial N}{\partial x}$

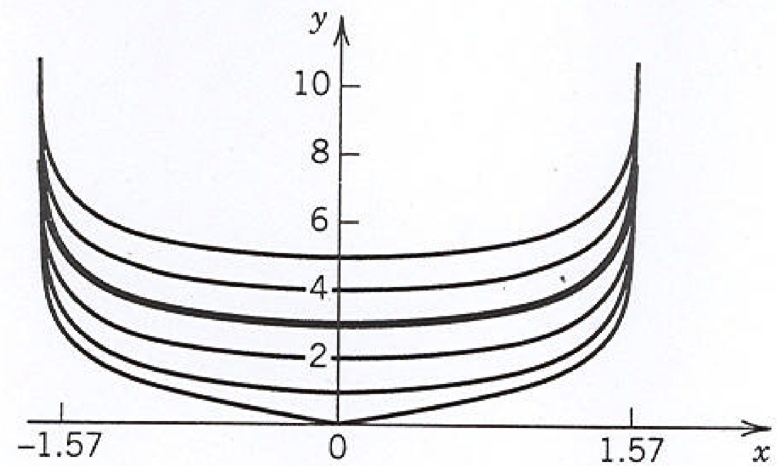
Integration: $u = \int (\sin x \cosh y)\, dx + k(y) = -\cos x \cosh y + k(y)$

Determination of k: $\dfrac{\partial u}{\partial y} = -\cos x \sinh y + \dfrac{dk}{dy} = N = -\cos x \sinh y$

$\Rightarrow \dfrac{dk}{dy} = 0 \Rightarrow k = C$

$\Rightarrow u(x,y) = -\cos x \cosh y + C$ and since $du = 0 \Rightarrow u = C^{te}$ $\Rightarrow \cos x \cosh y = C$

IC: $\cos 0 \cosh 3 = C \simeq 10.07 \Rightarrow \cos x \cosh y = 10.07$



Reduction to exact form using integrating factors

If $P(x,y)\,dx + Q(x,y)\,dy = 0$ not EDE

Multiplication by **integrating factor** $F(x,y)$ makes DE exact $\Rightarrow FPdx + FQdy = 0$

Note: may exist more than one integrating factor

Condition: $\frac{\partial}{\partial y} FP = \frac{\partial}{\partial x} FQ \Rightarrow F_y P + F P_y = F_x Q + F Q_x$

Easy to solve if $F = F(x) \Rightarrow F_y = 0$ and $F_x = \frac{dF}{dx}$

$$\Rightarrow FP_y = \frac{dF}{dx} Q + FQ_x \Rightarrow \frac{1}{F}\frac{dF}{dx} = \frac{1}{Q}\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right] = R(x)$$

Th. 1 if right side depends only on x then $F(x) = \exp\left[\int R(x)\,dx\right]$

Note: similarly we could have worked with $F(y)$ *for convenience sake*

Ex. 1 $2\sin(y^2)\,dx + (xy\cos y^2)\,dy = 0$ with IC: $y(2) = \sqrt{p/2}$

Verification: $\frac{\partial M}{\partial y} = 4y\cos y^2 \neq \frac{\partial N}{\partial x} = y\cos y^2$ ➔ not exact DE

We have $P = 2\sin y^2 \Rightarrow P_y = 4y\cos y^2$ and $Q = xy\cos y^2 \Rightarrow Q_x = y\cos y^2$

Since $R(x) = \frac{1}{Q}\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right] \Rightarrow R(x) = \frac{1}{xy\cos y^2}\left[4y\cos y^2 - y\cos y^2\right] = \frac{3y\cos y^2}{xy\cos y^2} = \frac{3}{x}$

Therefore $F(x) = \exp\left[\int R(x)\,dx\right] \Rightarrow F(x) = \exp\left[\int \frac{3}{x}dx\right] = \exp[3\ln x] = x^3$

Multiplication by integrating factor: $2x^3 \sin\left(y^2\right)dx + \left(x^4 y \cos y^2\right)dy = 0$

Verification: $\dfrac{\partial M}{\partial y} = 4x^3 y \cos y^2 = \dfrac{\partial N}{\partial x}$, exact

Solution: $u = \int 2x^3 \sin\left(y^2\right)dx + k\,(y) = \dfrac{1}{2}x^4 \sin y^2 + k\left(y\right)$

$\Rightarrow u_y = x^4 y \cos y^2 + k'\left(y\right) = x^4 y \cos y^2 \Rightarrow k'\left(y\right) = 0 \Rightarrow k\left(y\right) = C_1$

$\Rightarrow u\left(x, y\right) = \dfrac{1}{2}x^4 \sin y^2 = C$

IC: $\dfrac{1}{2}2^4 \sin\dfrac{p}{2} = 8 = C \Rightarrow u\left(x, y\right) = \dfrac{1}{2}x^4 \sin y^2 = 8$

Linear DE

Linear DE: $y' + p(x)y = r(x)$

Homogeneous linear DE: $r(x) = 0$

Solution: **separation of variables**

If **non homogeneous**: $(py - r)dx + dy = 0 \Rightarrow Pdx + Qdy = 0$

Integrating factor: $\frac{1}{F}\frac{dF}{dx} = \frac{1}{Q}\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right] = p \Rightarrow F(x) = e^{\int pdx}$

$$\Rightarrow e^{\int pdx}(y' + py) = \left(e^{\int pdx}y\right)' = e^{\int pdx}r$$

$$\Rightarrow e^{\int pdx}y = \int e^{\int pdx}rdx + C$$

Putting $h = \int pdx$ a general solution is: $y(x) = e^{-h}\left[\int e^{h}rdx + C\right]$

Ex. 1 $y' - y = e^{2x}$

➔ $p = -1$ and $r = e^{2x} \Rightarrow h = \int pdx = -x$

$$\Rightarrow e^{-x}y(x) = \int e^{-x}e^{2x}dx + C$$

$$\Rightarrow y(x) = e^{x}\left\{\int e^{-x}e^{2x}dx + C\right\} = e^{x}\left\{e^{x} + C\right\} = Ce^{x} + e^{2x}$$

More simply ➔ using integration factor = e^{-x}

$$\Rightarrow (y' - y)e^{-x} = e^{-x}e^{2x} \Rightarrow \left(ye^{-x}\right)' = e^{x} \Rightarrow ye^{-x} = e^{x} + C \Rightarrow y = e^{2x} + Ce^{x}$$

Ex. 2 $y' + 2y = e^x(3\sin 2x + 2\cos 2x)$

➔ $p = 2 \Rightarrow h = 2x$

$\Rightarrow y = e^{-2x}\left[\int e^{2x}e^x(3\sin 2x + 2\cos 2x)dx + C\right] = e^{-2x}\left[e^{3x}\sin 2x + C\right] = Ce^{-2x} + e^x \sin 2x$

Integration: $\Rightarrow \int e^{3x}(3\sin 2x + 2\cos 2x)dx = \int e^{3x}3\sin 2x dx + \int e^{3x}2\cos 2x dx$

By part: $\int u dv = uv - \int v du$

The second integral yields: $u = e^{3x} \Rightarrow du = 3e^{3x}$ and $dv = 2\cos 2x \Rightarrow v = \sin 2x$

$\Rightarrow \int e^{3x}3\sin 2x dx + \int e^{3x}2\cos 2x dx = \int e^{3x}3\sin 2x dx + e^{3x}\sin 2x - \int e^{3x}3\sin 2x dx = e^{3x}\sin 2x$

$\Rightarrow y = e^{-2x}\left[e^{3x}\sin 2x + C\right] = Ce^{-2x} + e^x \sin 2x$

Ex. 3 $y' + \tan x = \sin 2x$ with IC: $y(0) = 1$

$p = \tan x \Rightarrow h = \int \tan x dx = \ln|\sec x| \Rightarrow e^h = \sec x \Rightarrow e^{-h} = \cos x$

$r = \sin 2x = 2\sin x \cos x$

$\Rightarrow e^h r = \sec x \cdot 2\sin x \cos x = 2\sin x$

$\Rightarrow y = \cos x\left[2\int \sin x dx + C\right] = C\cos x - 2\cos^2 x$

IC: $\Rightarrow 1 = C \cdot 1 - 2 \cdot 1^2 \Rightarrow C = 3$

$\Rightarrow y = 3\cos x - 2\cos^2 x$

Reduction to linear form

Bernoulli equation: $y' + p(x)y = g(x)y^a$

Where a is any real number ➔ for $a = 0$ or $a = 1$ yields linear DE

Transformation: $u(x) = [y(x)]^{1-a}$

$\Rightarrow u' = (1-a)y^{-a}y' = (1-a)y^{-a}(gy^a - py) \Rightarrow u' = (1-a)(g - py^{1-a})$

$\Rightarrow u' + (1-a)pu = (1-a)g$

Ex. 1 Verhulst equation (logistic population model)

$y' - Ay = -By^2$ where A and B are positive constant

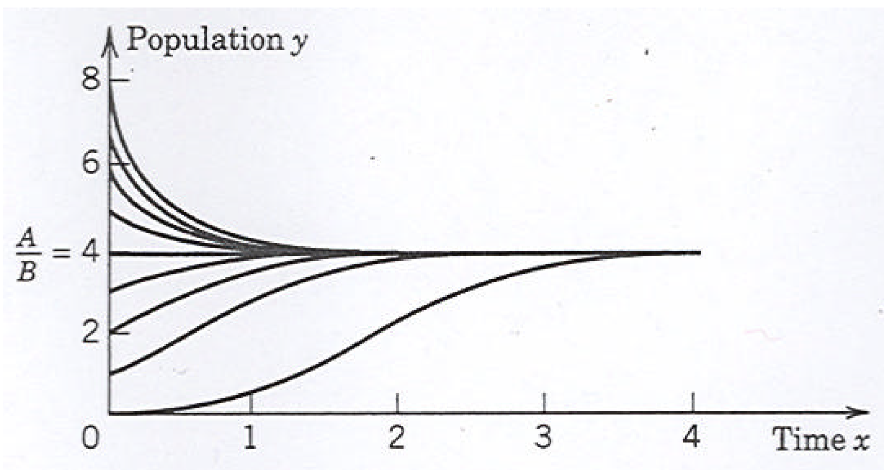
Bernoulli form with $a = 2 \Rightarrow u = y^{-1}$

$\Rightarrow u' = -y^{-2}y' = -y^{-2}\left(-By^2 + Ay\right) = B - Ay^{-1} \Rightarrow u' + uA = B$, which is linear

Using previous method: $p = A \qquad h = Ax \qquad r = B$

$$\Rightarrow u = e^{-Ax}\left[\int Be^{Ax}dx + C\right] = e^{-Ax}\left[\frac{B}{A}e^{Ax} + C\right] = Ce^{-Ax} + \frac{B}{A}$$

General solution: $y = \dfrac{1}{u} = \dfrac{1}{\dfrac{B}{A} + Ce^{-Ax}}$



Interpretation: putting $x = t$ we find the logistic law of population growth

For $B = 0$ the population grows exponentially $y = \dfrac{1}{C}e^{At}$ (law of Malthus)

The breaking term $-By^2$ prevent the population to grow without bound

Initially small populations $0 < y(0) < \dfrac{A}{B}$ increases monotone to $\dfrac{A}{B}$ while large populations $y(0) > \dfrac{A}{B}$ decrease to same value

Autonomous DE

In the logistic equation, time, the independent variable, does not appear explicitly

➔ $y' = f(t,y)$ = $y' = f(y)$ = **autonomous** DE

Autonomous DE has constant solutions = **equilibrium solutions** (or equilibrium points), determined by zeros (**critical points**) of $f(y)$, because $f(y)=0 \Rightarrow y'=0 \Rightarrow y=C^{te}$

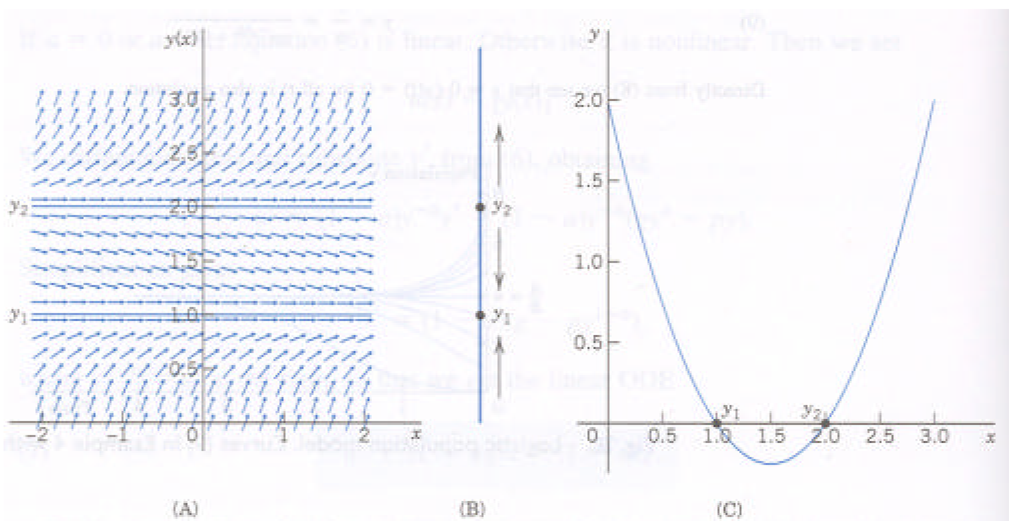
An equilibrium solution can be

- **stable** ➔ solutions close to it for some t remain close to it for all further t (ex $y=4$ in logistic equation)
- **unstable** ➔ solutions initially close to it do not remain close to it as t increases (ex. $y=0$ in logistic equation)

Ex. 1 phase line plot

$y' = (y-1)(y-2)$

- Stable equilibrium solution at $y_1 = 1$
- Unstable equilibrium at $y_2 = 2$



We can condense the direction field (A) to a phase line plot (C) giving y_1 and y_2 and direction of arrows (B) showing equilibrium

Orthogonal trajectories

Important class of problem in physics = find family of curves that intersect a given family of curves at right angle = orthogonal trajectories

Ex. **Isotherms**, **heat flows**, **equipotential curves**, **curves of steepest descent**, etc.

Step 1: find DE for which the given curves are solution curves $y' = f(x,y)$

Step 2: write down DE of orthogonal trajectories $y' = -\dfrac{1}{f(x,y)}$

Step 3: solve DE of orthogonal trajectories

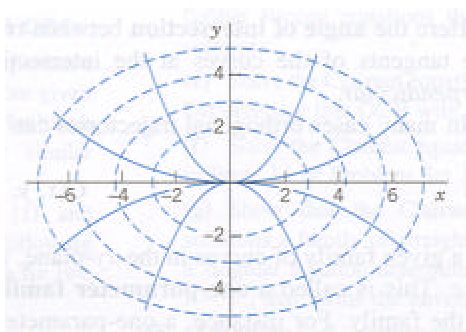
Ex. 1 one parameter family curves $F(x,y,c) = y - cx^2 = 0$ or $y = cx^2$ family of parabola

Step 1: $y' = 2cx$ …not good because we must eliminate parameter

$$\Rightarrow \frac{y}{x^2} = c \Rightarrow \frac{y'}{x^2} - 2\frac{y}{x^3} = 0 \Rightarrow y' = \frac{2y}{x}$$

Step 2: DE of orthogonal trajectories $y' = -\dfrac{x}{2y}$

Step 3: orthogonal trajectories $2y\,dy = -x\,dx \Rightarrow y^2 = -\dfrac{x^2}{2} + c^*$ one parameter family of ellipses $\dfrac{1}{2}x^2 + y^2 = c^*$



Existence and uniqueness of solutions

For DE, in most cases, general solutions exists

For an IVprb there exist 3 possibilities:

1) No solution
2) Only one solution
3) Infinity of solution

Problem of existence: conditions for **at least** one solution

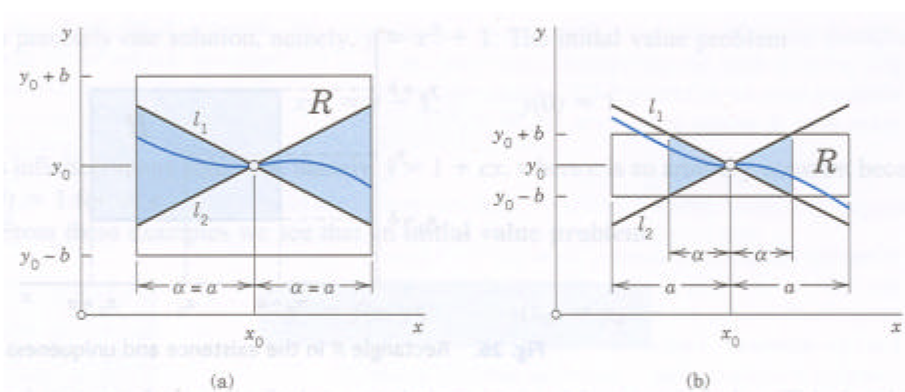
Problem of uniqueness: conditions for **at most** one solution

Th. 1 existence theorem

If $f(x,y)$ continuous at all points (x,y) in $R: |x-x_0| < a$ and $|y-y_0| < b$ and bounded in $R: |f(x,y)| \le K$ then the IVprb has at least one solution defined for all x in interval $|x-x_0| < \alpha$ where α smaller than two numbers a and $\frac{b}{K}$

Th. 2 Uniqueness theorem

If $f(x,y)$ and $\frac{\partial f}{\partial y}$ continuous for all (x,y) in R and bounded $f \le K$ and $\frac{\partial f}{\partial y} \le M$ then the IVprb has at most one solution $y(x)$ defined at least for all x in interval $|x-x_0| < \alpha$.



a) $\frac{b}{K} \ge a \Rightarrow \alpha = a$; b) $\frac{b}{K} < a \Rightarrow \alpha = \frac{b}{K}$, for larger or smaller x's nothing can be deduced

Linear DE of 2nd order

Many important applications in physics

- Theory typical of that of linear DE in general ➔ extension to higher order is straightforward

General form: $y'' + p(x)y' + q(x)y = r(x)$

Homogeneous: $y'' + p(x)y' + q(x)y = 0$

Solution: function $h(x)$ in interval $I: a < x < b$ and with continuous derivatives $h''(x)$ and $h'(x)$ in I

Ex. 1 $y'' - y = 0$ with $y = e^x$ or $y = e^{-x}$ as solutions

General solution = linear combination: $y = c_1 e^x + c_2 e^{-x}$

Superposition or linearity principle

Th. 1 Fundamental theorem for the homogeneous linear DE 2nd

For homogeneous linear DE of 2nd any linear combinations of 2 solutions (or sum and constant multiples) on an open interval are solutions

IVprb - General solution as basis

Solution: $y = c_1 y_1 + c_2 y_2$

IC ➔ 2 conditions: $y(x_0) = k_0$ and $y'(x_0) = k_1$

Ex. 2 $y'' - y = 0$ with IC: $y(0) = 4$ and $y'(0) = -2$

Applying IC to solution:

$y = c_1 e^x + c_2 e^{-x} \Rightarrow y(0) = c_1 + c_2 = 4$

$y' = c_1 e^x - c_2 e^{-x} \Rightarrow y'(0) = c_1 - c_2 = -2$

$\Rightarrow c_1 = 1$ and $c_2 = 3$

General solution: $y = e^x + 3e^{-x}$

A **general solution** of **homogeneous linear DE of 2nd order** is a solution of the type $y = c_1 y_1 + c_2 y_2$, where $y_1 \neq k y_2$

The two solutions form a **basis** ➔ $y_1(x)$ and $y_2(x)$ are **linearly independent** ➔ $k_1 y_1 + k_2 y_2 = 0$ implies $k_1 = k_2 = 0$

Ex. 3

$y'' - y = 0$ ➔ basis: $y = e^x$ and $y = e^{-x}$ ➔ solution: $y = c_1 e^x + c_2 e^{-x}$

$y'' + y = 0$ ➔ basis: $y = \cos x$ and $y = \sin x$ ➔ solution: $y = c_1 \cos x + c_2 \sin x$

Reduction of order

Method to find y_2 when y_1 is known $\Rightarrow y_1'' + py_1' + qy_1 = 0$

Set $y_2 = uy_1 \Rightarrow y_2' = u'y_1 + uy_1' \Rightarrow y_2'' = u''y_1 + 2u'y_1' + uy_1''$

Substitute in DE: $y_2'' + py_2' + qy_2 = 0$

$$\Rightarrow \left(u''y_1 + 2u'y_1' + uy_1''\right) + p\left(u'y_1 + uy_1'\right) + quy_1 = 0$$

$$\Rightarrow u''y_1 + u'\left(2y_1' + py_1\right) + u\left(y_1'' + py_1' + qy_1\right) = 0$$

$$\Rightarrow u'' + u'\left(\frac{2y_1' + py_1}{y_1}\right) = 0$$

Or $U' + \left(\dfrac{2y_1'}{y_1} + p\right)U = 0$

Separation of variables: $\dfrac{dU}{U} = -\left(\dfrac{2y_1'}{y_1} + p\right)dx \Rightarrow \ln|U| = -2\ln|y_1| - \int p\,dx \Rightarrow U = \dfrac{1}{y_1^2}e^{-\int p\,dx}$

Because $U = u'$ and $y_2 = uy_1 \Rightarrow y_2 = y_1\int U\,dx$

Since $\dfrac{y_2}{y_1} = u = \int U\,dx$, the ratio is not constant and the two solutions form a basis.

Ex. 4 $x^2y'' - xy' + y = 0$ or $y'' - \dfrac{y'}{x} + \dfrac{y}{x^2} = 0$

By inspection $y_1 = x$ since $y' = 1$ and $y'' = 0$

Then $p = -\dfrac{1}{x} \Rightarrow -\int p\,dx = \ln x$

And $U = \dfrac{1}{x^2}e^{\ln x} = \dfrac{1}{x} \Rightarrow y_2 = x\int U\,dx = x\ln x$

General solution: $y(x) = c_1x + c_2x\ln x$

A **particular solution** describes the unique behavior of a given physical system. If p, q and r are continuous function on I then there always exist a solution on I which is unique (no singular solution exists)

Homogeneous linear DE of 2nd order with constant coefficients

General form: $y'' + ay' + by = 0$

As a solution we try: $y = e^{lx} \Rightarrow y' = l e^{lx} \Rightarrow y'' = l^2 e^{lx}$

Replacing in the equation: $\left(l^2 + al + b\right)e^{lx} = 0$

Solution ➔ $\left(l^2 + al + b\right) = 0$ (**characteristic equation**)

Roots:
$$l_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right)$$
$$l_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right)$$

So the 2 **basis** are: $y_1 = \exp(l_1 x)$ and $y_2 = \exp(l_2 x)$

Depending on the sign of the discriminant $a^2 - 4b$ ➔ 3 solutions possible:

1) > 0 ➔ 2 distinct real roots
2) $= 0$ ➔ 1 real double root
3) < 0 ➔ 2 complex conjugate roots

Case 1 ➔ 2 distinct real roots $\Rightarrow y = c_1 e^{l_1 x} + c_2 e^{l_2 x}$

Ex. 1 DE: $y'' + y' - 2y = 0$ IC: $y(0) = 4$ and $y'(0) = -5$

$$\Rightarrow l_1 = \frac{1}{2}\left(-1 + \sqrt{9}\right) = 1 \quad l_2 = \frac{1}{2}\left(-1 - \sqrt{9}\right) = -2$$

$$\Rightarrow y = c_1 e^{x} + c_2 e^{-2x} \Rightarrow y' = c_1 e^{x} - 2c_2 e^{-2x}$$

IC:
$$y(0) = c_1 + c_2 = 4$$
$$y'(0) = c_1 - 2c_2 = -5$$

$$\Rightarrow c_1 = 1 \text{ and } c_2 = 3 \Rightarrow y = e^{x} + 3e^{-2x}$$

Case 2 ➔ real double root

$l = l_1 = l_2 = -\frac{a}{2}$ ➔ A first solution $y_1 = e^{-\frac{a}{2}x}$

Second solution: $y_2 = uy_1 \Rightarrow y'_2 = u'y_1 + uy'_1 \Rightarrow y''_2 = u''y_1 + 2u'y'_1 + uy''_1$

$\Rightarrow (u''y_1 + 2u'y'_1 + uy''_1) + a(u'y_1 + uy'_1) + buy_1 = 0$, which reduces to

$u''y_1 + u'(2y'_1 + ay_1) = 0$

But since $y' = -\frac{a}{2}y_1 \Rightarrow u''y_1 = 0 \Rightarrow u'' = 0 \Rightarrow u = c_1x + c_2$

Possible second solution is $y_2 = xy_1$ and general solution is $y = (c_1 + c_2x)e^{-\frac{a}{2}x}$

Ex. 2 $y'' + 8y' + 16y = 0$

Characteristic equation: $l^2 + 8l + 16 = 0$ ➔ double root: $l = -4$

General solution: $y = (c_1 + c_2x)e^{-4x}$

Ex. 3 IVprob: DE: $y'' - 4y' + 4y = 0$ CI: $y(0) = 3$ and $y'(0) = 1$

Characteristic equation: $l^2 - 4l + 4 = (l - 2)^2 \Rightarrow l = 2$

General solution: $y = (c_1 + c_2x)e^{2x} \Rightarrow y' = c_2e^{2x} + 2(c_1 + c_2x)e^{2x}$

IC: $y(0) = c_1 = 3$ $y'(0) = c_2 + 2c_1 = 1 \Rightarrow c_2 = -5$

Particular solution: $y = (3 - 5x)e^{2x}$

Case 3: $a^2 - 4b < 0$ ➔ **complex roots**

Ex. 1 $y'' + y = 0 \Rightarrow l^2 + 1 = 0 \Rightarrow l^2 = -1 \Rightarrow l = \pm\sqrt{-1} = \pm i$

Basis: e^{ix} and e^{-ix}

Using Euler formula:
$$e^{ix} = \cos x + i\sin x$$
$$e^{-ix} = \cos x - i\sin x$$

$$\Rightarrow \cos x = \frac{1}{2}\left[e^{ix} + e^{-ix}\right]$$

$$\Rightarrow \sin x = \frac{1}{2i}\left[e^{ix} - e^{-ix}\right]$$

$\Rightarrow \cos x$ and $\sin x$ are also solutions

Complex exponential function

$$e^{z} = e^{s+it} = e^{s}e^{it} = e^{s}\left(\cos t + i\sin t\right)$$

Obtained from **Maclaurin series** of e^{x}

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots$$

$$\Rightarrow e^{it} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right)$$

Since $\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$ and $\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$

$$\Rightarrow e^{it} = \cos t + i\sin t$$

Case 3 (continue): complex roots

To make radicand positive:

$$l_1 = -\frac{1}{2}a + \frac{1}{2}\sqrt{a^2 - 4b} = -\frac{1}{2}a + \frac{1}{2}i\sqrt{4b - a^2} = -\frac{1}{2}a + i\sqrt{b - \frac{1}{4}a^2}$$

$$l_1 = -\frac{1}{2}a + iw \text{ and } l_2 = -\frac{1}{2}a - iw \text{ where } w = \sqrt{b - \frac{1}{4}a^2}$$

$$\Rightarrow e^{l_1 x} = e^{-\frac{a}{2}x + iwx} = e^{-\frac{a}{2}x}\left(\cos wx + i\sin wx\right)$$

$$\Rightarrow e^{l_2 x} = e^{-\frac{a}{2}x - iwx} = e^{-\frac{a}{2}x}\left(\cos wx - i\sin wx\right)$$

Adding and dividing by 2: $\Rightarrow y_1 = e^{-\frac{a}{2}x}\cos wx$

Subtracting and dividing by $2i$: $\Rightarrow y_2 = e^{-\frac{a}{2}x}\sin wx$

General solution: $y = e^{-\frac{a}{2}x}\left(A\cos wx + B\sin wx\right)$

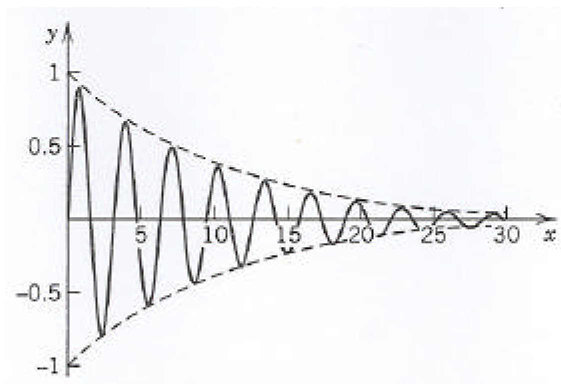
Ex. 2 DE: $y'' + 0.2y' + 4.01y = 0$

IC: $y(0) = 0 \quad y'(0) = 2$

Characteristic equation:
$l^2 + 0.2l + 4.01 = 0$

Complex roots: $-0.1 \pm 2i \Rightarrow w = 2$

General solution:
$y = e^{-0.1x}\left(A\cos 2x + B\sin 2x\right)$



IC: $y(0) = A = 0$

$y' = Be^{-0.1x}\left(-0.1\sin 2x + 2\cos 2x\right)$
$\Rightarrow y'(0) = 2B = 2 \Rightarrow B = 1$

Particular solution: $y = e^{-0.1x}\sin 2x$

***Interpretation*: damped vibration** ➔ amplitude of oscillation decreases exponentially

Summary of cases 1 to 3

Case	Roots	Basis	General solution
1	Distinct real l_1, l_2	$e^{l_1 x}, e^{l_2 x}$	$y = c_1 e^{l_1 x} + c_2 e^{l_2 x}$
2	Real double $l = -\frac{1}{2}a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + xc_2) e^{-ax/2}$
3	Complex conjugate $l_1 = -\frac{1}{2}a + iw$ $l_1 = -\frac{1}{2}a - iw$	$e^{-ax/2} \cos wx$ $e^{-ax/2} \sin wx$	$y = e^{-ax/2} (A\cos wx + B\sin wx)$

Note: in mechanics or electrical circuits, these 3 cases correspond to 3 different forms of motions or flows current

Differential operator

The differential can be seen as a transformation writing DE as $Dy = y'$ ➔ **differential operator** D that operates on y

Ex. 1 $Dx^2 = 2x$ or $D(\sin x) = \cos x$

Higher derivative: $D(Dy) = D(y') = y''$ or $D^2 y$

Second order DE operator: $L = P(D) = D^2 + aD + b$

$\Rightarrow L(y) = (D^2 + aD + b)\, y = y'' + ay' + by$

The operator is linear: $L[a\,y + b\,w] = a\,L(y) + b\,L(w)$

Homogeneous differential equation: $L(y) = 0$

Since $D(e^{lx}) = l\,e^{lx} \Rightarrow D^2(e^{lx}) = l^2 e^{lx}$

$P(D)[e^{lx}] = (l^2 + a\,l + b)\,e^{lx} = P(l)\,e^{lx}$

Since e^{lx} is solution $P(l) = 0$

Case 1: $P(l)$ ➔ 2 different real roots

Case 2: $P(l)$ ➔ double real root ➔ needs second independent solution

Start with $P(D)\left[e^{lx}\right]=\left(l^2+al+b\right)e^{lx}=P(l)e^{lx}$

Differentiating: $\left(P(D)\left[e^{lx}\right]\right)'=P'(l)e^{lx}+P(l)xe^{lx}=P(D)\left[xe^{lx}\right]$

Because $P'=\dfrac{dP}{dl}\Rightarrow P'(l)e^{lx}+P(l)xe^{lx}=(2l+a)e^{lx}+\left(l^2+al+b\right)xe^{lx}$

$$=\left(2l+l^2x\right)e^{lx}+a(1+lx)e^{lx}+bxe^{lx}=\left(D^2+aD+b\right)xe^{lx}$$

For double root: $l=-\dfrac{1}{2}a\Rightarrow P(l)=P'(l)=0\Rightarrow P(D)\left(xe^{lx}\right)=0$

$\Rightarrow xe^{lx}$ is second solution

Since $P(l)$ is a polynomial $\Rightarrow P(D)$ is **polynomial operator**

Ex. 2 Factorize $P(D)=D^2+D-6$ and solve $P(D)[y]=0$

$D^2+D-6=(D+3)(D-2)$

Since $(D-2)y=y'-2y\Rightarrow(D+3)(D-2)y=(D+3)(y'-2y)=y''+y'-6y$

Similarly: $(D+3)y=y'+3y\Rightarrow(D-2)(D+3)y=(D-2)(y'+3y)=y''+y'-6y$

Solutions: $(D+3y)=0\Rightarrow y=e^{-3x}$ and $(D-2y)=0\Rightarrow y=e^{+2x}$

Modeling: free oscillation (mass-spring system)

Physical assumptions: mass of body >> mass of spring

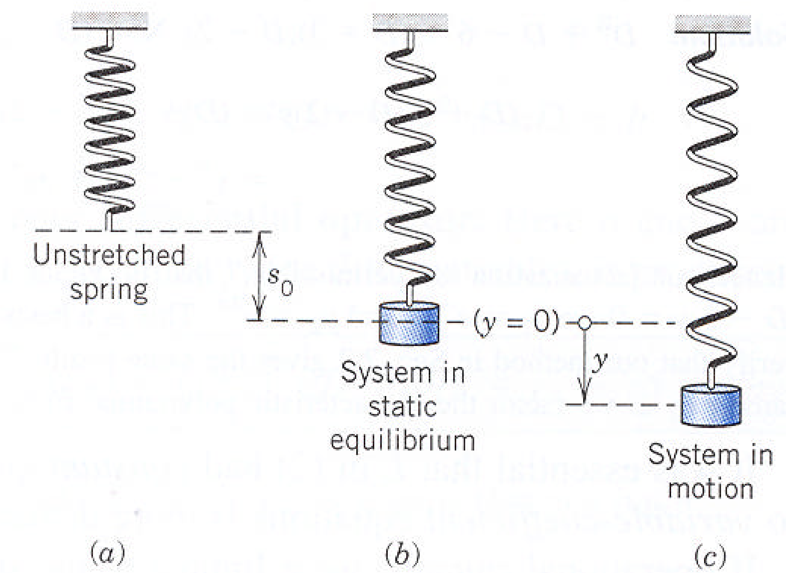
Orientation: positive displacements downward

Hooke's law: $F_0 = -ks_0$ where $k > 0$ **spring modulus** and s_0 **stretch** ➔ the smaller k the larger s_0

Assume **static equilibrium**: F_0 balance the weight $W = mg$

Applying Newton law: $F_0 + W = -ks_0 + mg = 0$

Putting $y = 0$ at position of equilibrium ➔ y measures displacement of body from equilibrium position



Experiment: pulling weight downward ➔ **restoring force** $F_1 = -ky$

Damping: dissipation of energy of mechanical system

Ignoring damping: $my'' + ky = 0 \Rightarrow y'' + \frac{k}{m} y = 0$

Solution: $y(t) = A\cos w_0 t + B\sin w_0 t$ where $w_0 = \sqrt{\frac{k}{m}}$

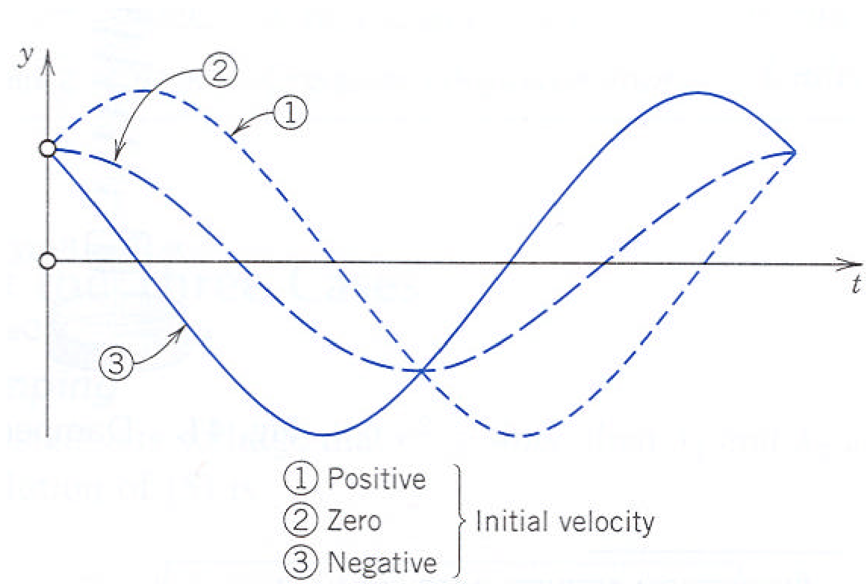
➔ **Harmonic oscillation**

Putting $B = \sin d$ and $A = \cos d$ $\Rightarrow C = \sqrt{A^2 + B^2}$ and $\tan d = \frac{B}{A}$

Solution transformed into $y(t) = C\cos(w_0 t - d)$

Period: $\frac{2p}{w_0}$ **Frequency**: $\frac{w_0}{2p}$ (Hertz)

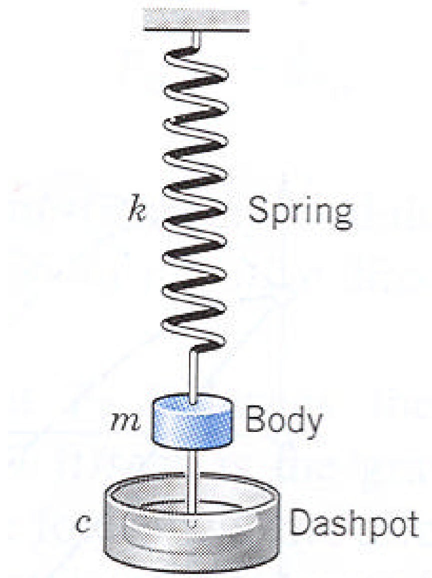
3 possible cases depending on initial velocity: $y' = 0$ or $y' = \pm V_0$



Including damping

➔ **damping force** $\propto y' = \frac{dy}{dt} \Rightarrow F_2 = -cy'$

Newton's law: $my'' + cy' + ky = y'' + \frac{c}{m}y' + \frac{k}{m}y = 0$



Characteristic equation: $l^2 + \frac{c}{m}l + \frac{k}{m} = 0$

Roots: $l_{1,2} = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4mk}$

Putting $a = \frac{c}{2m}$ and $b = \frac{1}{2m}\sqrt{c^2 - 4mk}$

$\Rightarrow l_1 = -a + b$ and $l_2 = -a - b$

Formal solution depends on amount of damping:

Case 1: **over-damping** $\Rightarrow c^2 > 4km$

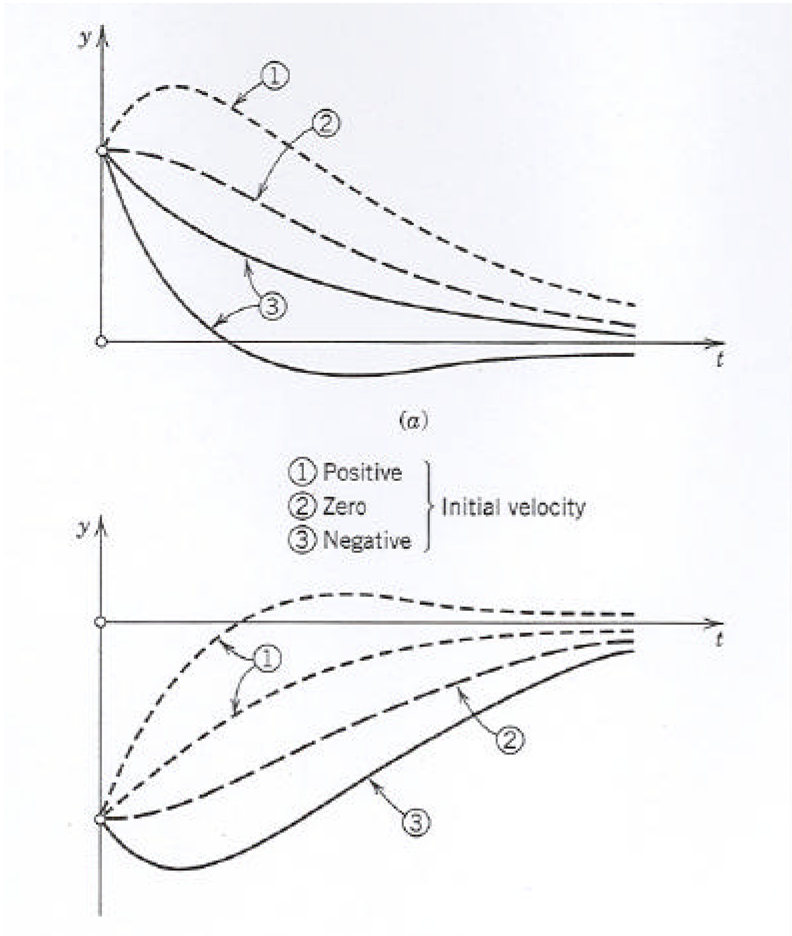
Case 2: **critical damping** $\Rightarrow c^2 = 4km$

Case 3: **under-damping** $\Rightarrow c^2 < 4km$

Solution in case 1 **(Over-damping):**

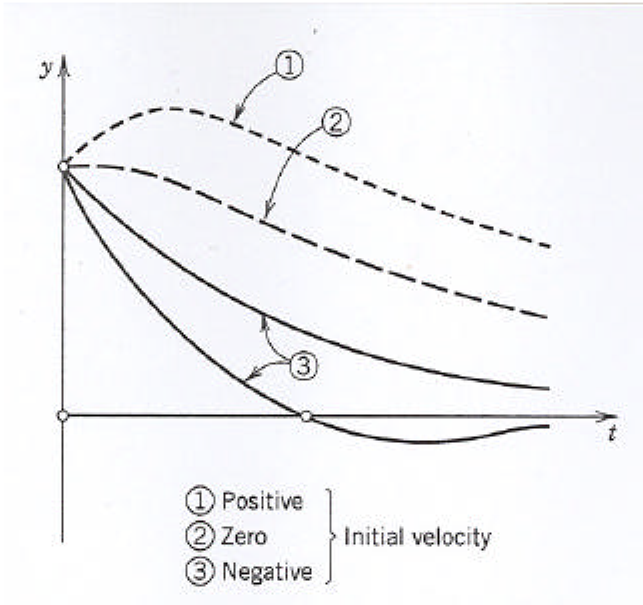
$y(t) = c_1 e^{-(a-b)t} + c_2 e^{-(a+b)t}$

Both terms approach zero as $t \to \infty \Rightarrow$ no oscillation



***Solution in case 2* (Critical damping)**: $y(t) = (c_1 + c_2 t)\, e^{-at}$

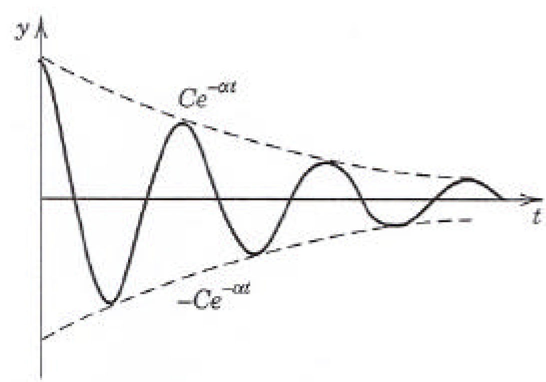
Term $(c_1 + c_2 t)$ allows one zero ➔ system pass at most by $y = 0$ position



***Solution in case 3* (Under-damping)**:

$y(t) = e^{-at}\left(A\cos w^* t + B\sin w^* t\right) = Ce^{-at}\cos\left(w^* t - d\right)$

Where $w^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$



Amplitude varies between curves $y = \pm Ce^{-at}$.

The smaller is c, the higher the frequency $\frac{w^*}{2p}$. At the limit $c \to 0$ $w^* \to w_0$

Euler-Cauchy equation

General form: $x^2 y'' + axy' + by = 0$ where a and b are constant

Trying the solution: $y = x^m \Rightarrow y' = mx^{m-1} \Rightarrow y'' = m(m-1)x^{m-2}$

$$\Rightarrow x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0$$
$$\Rightarrow m(m-1)x^m + amx^m + bx^m = 0$$

Dropping x^m factor $\Rightarrow m^2 + (a-1)m + b = 0$

Case 1: 2 distinct real roots $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$

➔ General solution: $y = c_1 x^{m_1} + c_2 x^{m_2}$

Ex. 1 $x^2 y'' - 2.5xy' - 2.0y = 0$

$$\Rightarrow m^2 - 3.5m - 2.0 = 0$$

$$\Rightarrow m_1 = -0.5 \text{ and } m_2 = 4$$

General solution: $y = \dfrac{c_1}{\sqrt{x}} + c_2 x^4$

Case 2: 1 double root $\frac{1}{2}(1-a)$

$$\left[m-\frac{1}{2}(1-a)\right]^2 = m^2+(a-1)m+b \text{ where } b=\frac{1}{4}(1-a)^2$$

First solution: $y_1 = x^{(1-a)/2}$

➔ for second solution, substituting $y_2 = uy_1$ in $x^2 y_1'' + axy_1' + by_1 = 0$

$$\Rightarrow x^2\left(u''y_1 + 2u'y_1' + uy_1''\right) + ax\left(u'y_1 + uy_1'\right) + buy_1 = 0$$

$$\Rightarrow u''x^2 y_1 + u'x\left(2xy_1' + ay_1\right) + u\left(x^2 y_1'' + axy_1' + by_1\right) = 0$$

Since last term is zero and $y_1' = \frac{1}{2}(1-a)\,x^{-1}x^{(1-a)/2} = \frac{1}{2}(1-a)\,x^{-1}y_1 \Rightarrow 2xy_1' + ay_1 = y_1$

$$\Rightarrow \left(u''x^2 + u'x\right)y_1 = 0 \Rightarrow \frac{u''}{u'} = -\frac{1}{x} \Rightarrow \ln|u'| = -\ln x \Rightarrow u' = \frac{1}{x} \Rightarrow u = \ln x$$

Second solution: $y_2 = y_1 \ln x$

General solution: $y = \left(c_1 + c_2 \ln x\right)x^{(1-a)/2}$

Ex. 2 $x^2 y'' - 3xy' + 4y = 0$

$\Rightarrow \frac{1}{2}(1-a) = 2 \Rightarrow m = 2$, and solution $y = \left(c_1 + c_2 \ln x\right)x^2$

Case 3 complex conjugate roots:

$m_1 = m + in$ and $m_2 = m - in$

$\Rightarrow x^{in} = \left(e^{\ln x}\right)^{in} = e^{in \ln x}$

Using Euler formula:

$x^{m_1} = x^m x^{in} = x^m e^{in \ln x} = x^m \left(\cos(n \ln x) + i \sin(n \ln x)\right)$

$x^{m_2} = x^m x^{-in} = x^m e^{-in \ln x} = x^m \left(\cos(n \ln x) - i \sin(n \ln x)\right)$

Usual transformation yields real solutions:

$y_1 = x^m \cos(n \ln x)$ and $y_2 = x^m \sin(n \ln x)$

Ex. 3

$x^2 y'' + 7xy' + 13y = 0$

$\Rightarrow m^2 + 6m + 13 = 0 \Rightarrow m_{1,2} = -3 \pm \sqrt{9-13} = -3 \pm 2i$

General solution: $y = x^{-3}\left[A\cos(2\ln x) + B\sin(2\ln x)\right]$

Existence and Uniqueness theory: Wronskian

Homogeneous linear DE of 2^{nd}: $y'' + p(x)y' + q(x)y = 0$

General solution: $y = c_1y_1 + c_2y_2$

IVprob: $y(x_0) = K_0$ and $y'(x_0) = K_1$

Th. 1 Existence and uniqueness for IVprob

If $p(x)$ and $q(x)$ continuous on I and x_0 in I then exist solution $y(x)$ on I

Linear independent solutions – **Wronskian**

Linear independence: $k_1y_1(x) + k_2y_2(x) = 0 \Rightarrow k_1 = k_2 = 0$

Wronskian determinant: $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$

Th. 2 linear dependence and independence

Linear dependent $\Rightarrow W = 0$ for $x = x_0$

Linear independent $\Rightarrow W \neq 0$

Th. 3 general solution

If $p(x)$ and $q(x)$ continuous on open interval I $\Rightarrow$ exists general solution

Th. 4 Form of general solution

Under conditions Th. 3, general solution $Y(x) = c_1y_1(x) + c_2y_2(x)$ where y_1 and y_2 are basis and coefficients are suitable constants.

No singular solution exists.

Nonhomogeneous DE of 2nd

General form: $y'' + p(x)y' + q(x)y = r(x)$

Th. 1 Relation between nonhomogeneous DE and homogeneous DE

a) The **difference** of 2 solutions of an nonhomogeneous DE on some open interval I is a solution of the equivalent homogenous DE on I

b) The **sum** of a solution of a nonhomogeneous DE on I and of a solution of the equivalent homogenous DE on I is a solution of the nonhomogeneous DE

General solution of nonhomogeneous DE on open interval I is of the form:
$y(x) = y_h(x) + y_p(x)$

Where $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$ = general solution of homogeneous DE and $y_p(x)$ = any solution of nonhomogeneous DE without arbitrary constants

Particular solution: c_1 and c_2 are values defined in $y_h(x)$

A general solution includes all solutions

If coefficients of nonhomogeneous DE of 2^{nd} and $r(x)$ are continuous in I

➔ Solution exists because both $y_h(x)$ and $y_p(x)$ exist on I

Once $y_p(x)$ established, IVprb solution is unique

Th. 2 Suppose that coefficient and $r(x)$ continuous on I then every solution of nonhomogeneous DE is obtained by assigning suitable values to arbitrary constants in general solution of nonhomogeneous DE

Practical conclusion: to solve a nonhomogeneous DE of 2^{nd} order or IVprb:

1) Solve the homogenous DE

2) Find particular solution $y_p(x)$

Ex. 1 DE: $y'' + 2y' + 101y = 10.4e^x$ IC: $y(0) = 1.1$ and $y'(0) = -0.9$

Step 1 – solve homogenous DE

$l^2 + 2l + 101 = 0 \Rightarrow l_{1,2} = -1 \pm 10i$

$\Rightarrow y_h = e^{-x}\left(A\cos 10x + B\sin 10x\right)$

Step 2 – determine particular solution of nonhomogeneous DE

We try $y_p = Ce^x$

$\Rightarrow (1 + 2 + 101)Ce^x = 10.4e^x$

$\Rightarrow C = 0.1$

General solution: $y = y_h + y_p = e^{-x}\left(A\cos 10x + B\sin 10x\right) + 0.1e^x$

Step 3 – particular solution

$y(0) = A + 0.1 = 1.1 \Rightarrow A = 1$

$y' = e^{-x}\left(-\cos 10x - B\sin 10x - 10\sin 10x + 10B\cos 10x\right) + 0.1e^x$

$\Rightarrow y'(0) = -1 + 10B + 0.1 = -0.9 \Rightarrow B = 0$

Particular solution: $y = e^{-x}\cos 10x + 0.1e^x$

Solution by undetermined coefficients

Considering the form $y'' + ay' + by = r(x)$

Choose for y_p a form similar to $r(x)$ involving unknown coefficients to be determined by substituting in solution.

Rules:

A) Basic – If $r(x)$ is one of the function in table below choose y_p in second column of table and determine coefficients by substituting y_p + derivatives in DE

Terms in $r(x)$	Choice for y_p
ke^{gx}	Ce^{gx}
$kx^n \quad (n = 0,1,2,\ldots)$	$K_n x^n + K_{n-1}x^{n-1} + \ldots + K_1 x + K_0$
$k\cos wx$ $k\sin wx$	$K\cos wx + M\sin wx$
$ke^{ax}\cos wx$ $ke^{ax}\sin wx$	$e^{ax}(K\cos wx + M\sin wx)$

B) Modification rule – if a term in y_p happens to be a solution of homogeneous DE then multiply by x and x^2 (double root)

C) Sum rule – if $r(x)$ is a sum of a functions in table then use sum of y_p

Ex. 1 $y'' + 4y = 8x^2$

$\Rightarrow y_p = K_2 x^2 + K_1 x + K_0$
$\Rightarrow y_p'' = 2K_2$

Substituting: $2K_2 + 4\left(K_2 x^2 + K_1 x + K_0\right) = 8x^2$

Equating coefficients: $4K_2 = 8$, $4K_1 = 0$, $2K_2 + 4K_0 = 0$

$\Rightarrow K_2 = 2$, $K_1 = 0$ and $K_0 = -1$

$\Rightarrow y_p = 2x^2 - 1$

General solution: $y = y_h + y_p = A\cos 2x + B\sin 2x + 2x^2 - 1$

Ex. 2 $y'' - 3y' + 2y = e^x$

Characteristic equation: $l^2 - 3l + 2 = 0 \Rightarrow l_1 = 1$ and $l_2 = 2$ $\Rightarrow y_h = c_1 e^x + c_2 e^{2x}$

Choice for y_p cannot be Ce^x because this is solution of homogenous DE

Trying (rule B) $y_p = Cxe^x$

$\Rightarrow y_p' = C\left(e^x + xe^x\right)$ and $y_p'' = C\left(2e^x + xe^x\right)$

Substituting: $C(2+x)e^x - 3C(1+x)e^x + 2Cxe^x = e^x \Rightarrow C = -1$

General solution: $y = c_1 e^x + c_2 e^{2x} - xe^x$

Ex. 3 IVprb DE: $y'' + 2y' + y = (D+1)^2 y = e^{-x}$ $\quad y(0) = -1$ and $y'(0) = 1$

Characteristic equation: $(l^2 + 1)^2 = 0 \Rightarrow l = -1$ (double root)

$\Rightarrow y_h = (c_1 + c_2 x)\, e^{-x}$

Rule B (double root) $\Rightarrow y_p = Cx^2 e^{-x}$

$\Rightarrow y_p' = C(2x - x^2)e^{-x}$ $\qquad \Rightarrow y_p'' = C(2 - 4x + x^2)e^{-x}$

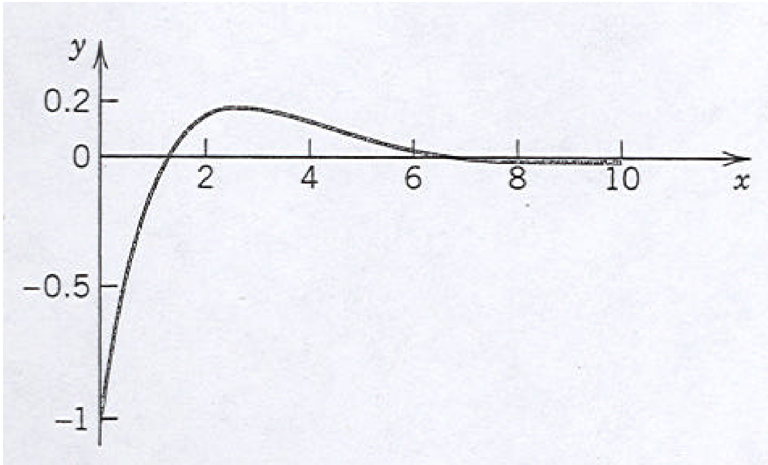
Substituting: x and x^2 terms drop out

$$\Rightarrow C(2 - 4x + x^2)e^{-x} + 2C(2x - x^2)e^{-x} + Cx^2e^{-x} = 2Ce^{-x} = e^{-x} \Rightarrow C = \frac{1}{2}$$

General solution: $y = (c_1 + c_2 x)e^{-x} + \frac{1}{2}x^2 e^{-x}$

IC: $y(0) = c_1 = -1$ $\quad$ and $y'(0) = c_2 - c_1 = 1 \Rightarrow c_2 = 0$

$$\Rightarrow y = \left(\frac{1}{2}x^2 - 1\right)e^{-x}$$



Ex. 4 DE: $y'' + 2y' + 5y = 1.25e^{0.5x} + 40\cos 4x - 55\sin 4x$

IC: $y(0) = 0.2$ $\qquad y'(0) = 60.1$

Characteristic equation: $l^2 + 2l + 5 = 0 \Rightarrow l_{1,2} = -1 \pm 2i$

$\Rightarrow y_h = e^{-x}\left(A\cos 2x + B\sin 2x\right)$

From the table: $y_p = Ce^{0.5x} + K\cos 4x + M\sin 4x$

$y_p' = 0.5Ce^{0.5x} - 4K\sin 4x + 4M\cos 4x$

$y_p'' = 0.25Ce^{0.5x} - 16K\cos 4x - 16M\sin 4x$

Substituting: $(0.25 + 1 + 5)Ce^{0.5x} + (-16K + 8M + 5K)\cos 4x + (-16M - 8K + 5M)\sin 4x$

Equating coefficients: $6.25C = 1.25 \Rightarrow C = 0.2$

$$\left.\begin{aligned} -11K + 8M &= 40 \\ -8K - 11M &= -55 \end{aligned}\right\} \Rightarrow K = 0 \text{ and } M = 5$$

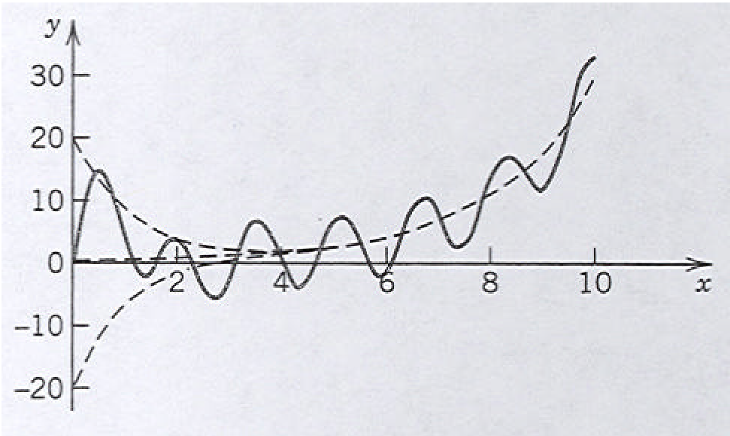
General solution: $y = e^{-x}\left(A\cos 2x + B\sin 2x\right) + 0.2e^{0.5x} + 5\sin 4x$

IC: $y(0) = A + 0.2 = 0.2 \Rightarrow A = 0$

$y' = e^{-x}\left(-B\sin 2x + 2B\cos 2x\right) + 0.1e^{0.5x} + 20\cos 4x$

$y'(0) = 2B + 0.1 + 2.0 = 60.1 \Rightarrow B = 20$

Particular solution: $y = 20e^{-x}\sin 2x + 0.2e^{0.5x} + 5\sin 4x$



Solution by variation of parameters

Standard form: $y'' + p(x)\,y' + q(x)\,y = r(x)$

With *p*, *q*, and *r* arbitrary variable functions continuous in interval I

Particular solution: $y_p(x) = -y_1 \int \frac{y_2 r}{W}\,dx + y_2 \int \frac{y_1 r}{W}\,dx$

Where y_1 and y_2 solutions of homogeneous equation and $W = y_1 y_2' - y_2 y_1'$

Ex. 1 $y'' + y = \sec x$

Solution to homogeneous equation: $y'' + y = 0 \Rightarrow y_1 = \cos x$ and $y_2 = \sin x$

$W = \cos x \cos x - \sin x(-\sin x) = 1$

$\Rightarrow y_p = -\cos x \int \sin x \sec x dx + \sin x \int \cos x \sec x dx = \cos x \ln|\cos x| + x \sin x$

General solution: $y = (c_1 + \ln|\cos x|)\cos x + (c_2 + x)\sin x$

Modeling: forced oscillations. Resonance

Free motion of mass on spring ➔ homogenous DE: $my'' + cy' + ky = 0$

Driving force (or **input**) $r(t)$ ➔ nonhomogeneous DE: $my'' + cy' + ky = r(t)$

Particular interest: $r(t) = F_0 \cos wt$

Solution = **output** or **response:** $y = y_h + y_p$

Method of underdetermined coefficients $\Rightarrow y_p(t) = a\cos wt + b\sin wt$

$$\Rightarrow y_p'(t) = -wa\sin wt + wb\cos wt$$

$$\Rightarrow y_p''(t) = -w^2 a\cos wt - w^2 b\sin wt$$

Substituting: $\left[\left(k - mw^2\right)a + wcb\right]\cos wt + \left[-wca + \left(k - mw^2\right)b\right]\sin wt = F_0 \cos wt$

$$\begin{cases} \left(k - mw^2\right)a + wcb = F_0 \\ -wca + \left(k - mw^2\right)b = 0 \end{cases}$$

Linear system of 2 algebraic equations in two unknowns
Solution by elimination (Chap 6)

$$a = F_0 \frac{k - mw^2}{\left(k - mw^2\right)^2 + w^2c^2} \quad \text{and} \quad b = F_0 \frac{wc}{\left(k - mw^2\right)^2 + w^2c^2}$$

Putting $w_0 = \sqrt{\dfrac{k}{m}}$ with $w_0 > 0$

$$a = F_0 \frac{m\left(w_0^2 - w^2\right)}{m^2\left(w_0^2 - w^2\right)^2 + w^2c^2} \quad \text{and} \quad b = F_0 \frac{wc}{m^2\left(w_0^2 - w^2\right)^2 + w^2c^2}$$

Case I: undamped forced oscillation

No damping $\Rightarrow c = 0$

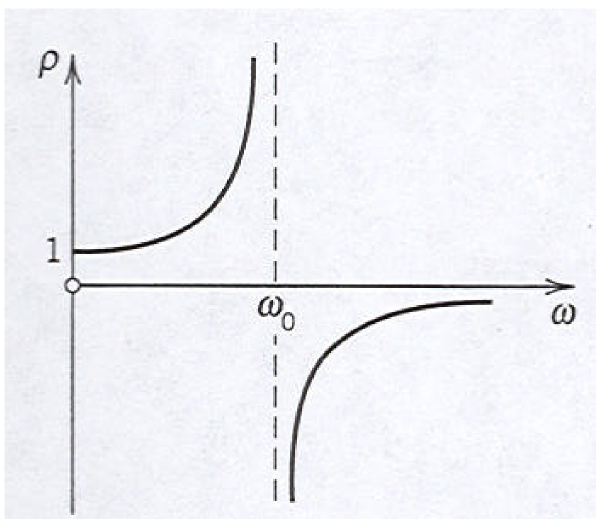
$$y_p(t) = \frac{F_0}{m\left(w_0^2 - w^2\right)}\cos w\,t = \frac{F_0}{k\left[1-\left(\frac{w}{w_0}\right)^2\right]}\cos w\,t$$

$$y = C\cos\left(w_0 t - d\right) + \frac{F_0}{k\left[1-\left(\frac{w}{w_0}\right)^2\right]}\cos wt$$

Interpretation: output = superposition of 2 harmonic oscillations, one with **natural frequency** $\frac{w_0}{2p}$ and the other with the frequency of the input $\frac{w}{2p}$

Maximum amplitude: $a_0 = \frac{F_0}{k}r$ where $r = \frac{1}{1-\left(\frac{w}{w_0}\right)^2}$ is the **resonance factor**

Ratio of amplitude of y_p and input: $\frac{r}{k} = \frac{a_0}{F_0}$

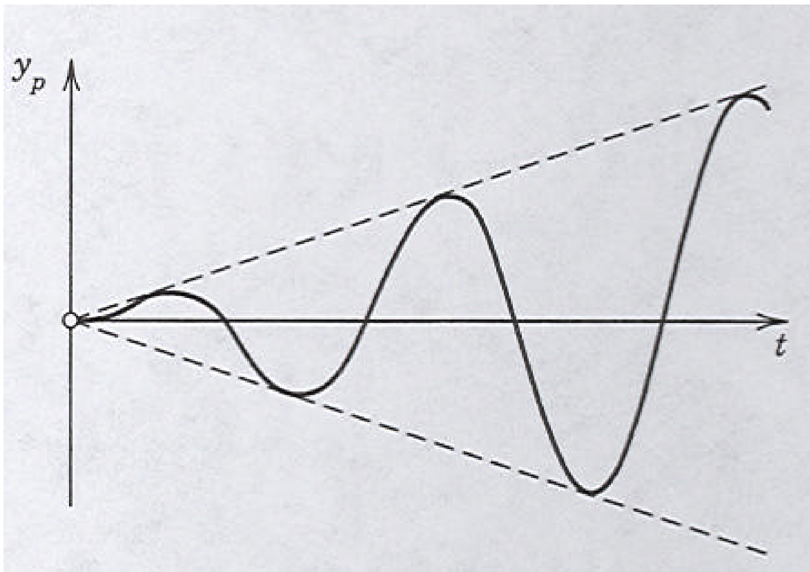


Resonance phenomenon:

For $w \to w_0 \Rightarrow r \to \infty \Rightarrow a_0 \to \infty$

In case of resonance DE: $y'' + w_0^2 y = \frac{F_0}{m}\cos w_0 t$

From modification rule: $y_p(t) = \frac{F_0}{2mw_0} t \sin w_0 t$ ➔ becomes larger and larger



Without damping, vibrations could destroy the system

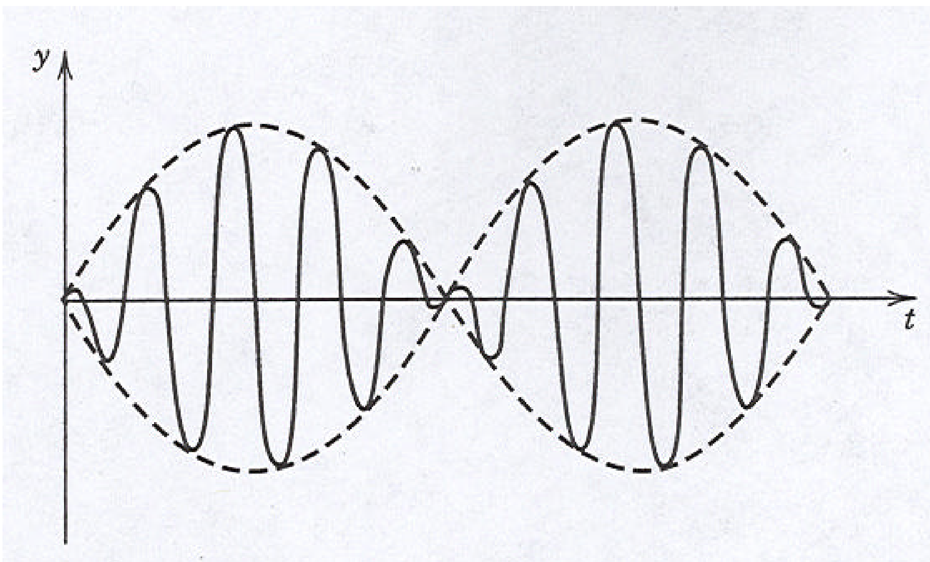
Beats:

If $w \neq w_0$ but very close, for IC: $y(0) = y'(0) = 0$

Particular solution: $y(t) = \dfrac{F_0}{m\left(w_0^2 - w^2\right)}\left(\cos wt - \cos w_0 t\right)$

Or $y(t) = \dfrac{2F_0}{m\left(w_0^2 - w^2\right)} \sin\left(\dfrac{w_0 + w}{2} t\right) \sin\left(\dfrac{w_0 - w}{2} t\right)$

Since $w_0 - w$ small, period of last sine becomes very large ➔ this produces the **beat phenomenon** (amplitude varies harmonically)



Case II: damped forced oscillation

Damping $\Rightarrow c > 0$

$$y_h(t) = e^{-at}\left(A\cos w^* t + b\sin w^* t\right) \text{ with } w^* = \frac{c}{2m}$$

The solution approach zero after a sufficiently long time

Solution of nonhomogeneous DE = **transient solution**: $y = y_h + y_p$

Approach steady state solution y_p after sufficiently long time ➔ output becomes harmonic oscillation

This is the normal behavior of real physical systems ➔ no damping is ideal case

Practical resonance: in damped case amplitude stay finite as $w \to w_0$, becomes maximum for some frequency $w(c)$ ➔ some input may destroy the system

Amplitude of y_p

$$y_p(t) = C^* \cos(wt - h) \text{ where } C^* = \sqrt{a^2 + b^2} = \frac{F_0}{\sqrt{m^2\left(w_0^2 - w^2\right)^2 + w^2c^2}}$$

and $\tan h = \dfrac{b}{a} = \dfrac{wc}{m\left(w_0^2 - w^2\right)}$

Maximum ➔ $\dfrac{dC^*}{dw} = 0 \Rightarrow \left[-2m^2\left(w_0^2 - w^2\right) + c^2\right]w = 0$

$\Rightarrow 2m^2\left(w_0^2 - w^2\right) = c^2$

For large damping $c^2 > 2m^2w_0^2 = 2mk$ ➔ no solution C^* decreases in monotone way as w increases

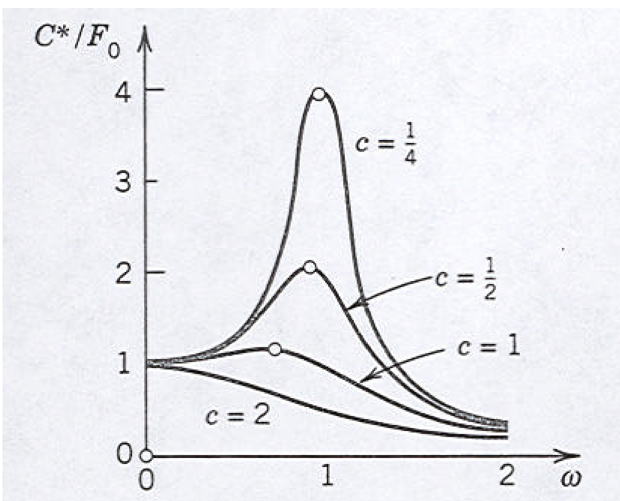
If $c^2 \le 2m^2 w_0^2$ ➔ one solution

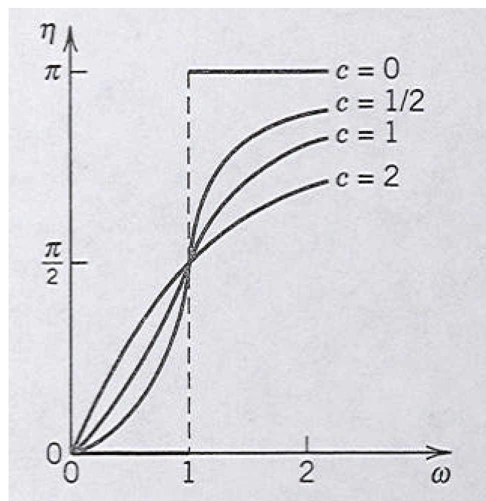
For $w = w_{\max} = \sqrt{w_0^2 - \dfrac{c^2}{2m^2}}$ increases as c decreases approaching w_0 as $c \to 0$

Amplitude $C^*\left(w_{\max}\right) = \dfrac{2mF_0}{c\sqrt{4m^2 w_0^2 - c^2}}$ always finite for $c > 0$

Since $C^*\left(w\right)$ increases as c decreases and approaches ∞ as $c \to 0$

Phase lag h **:** $\begin{cases} w < w_0 \Rightarrow h < \dfrac{p}{2} \\ w = w_0 \Rightarrow h = \dfrac{p}{2} \\ w > w_0 \Rightarrow h > \dfrac{p}{2} \end{cases}$





Note: Whereas DE of 2nd order have various applications, higher order DE occur much more rarely in physics (Ex. binding of elastic beams)

The theory is much similar to that of DE of 2nd order

2) Transformation $v = ay + bx + k$

Ex. 7 $(2x - 4y + 5)\, y' + x - 2y + 3 = 0$

The terms $2x - 4y$ and $x - 2y$ ➔ $v = x - 2y \Rightarrow 2y = x - v \Rightarrow y' = \frac{1}{2}(1 - v')$

Substitution: $\left[2x - 2(x - v) + 5\right]\frac{1}{2}(1 - v') + x - (x - v) + 3 = 0$

$$\Rightarrow [2v + 5]\frac{1}{2}(1 - v') + v + 3 = 0 \Rightarrow (2v + 5) - (2v + 5)v' = -2v - 6$$

$$\Rightarrow (2v + 5)v' = 4v + 11 \Rightarrow \left(\frac{4v + 10}{4v + 11}\right)dv = 2dx \Rightarrow \left(\frac{4v + 11 - 1}{4v + 11}\right)dv = 2dx$$

$$\Rightarrow \int\left(1 - \frac{1}{4v + 11}\right)dv = \int 2dx + C_1 \Rightarrow v - \frac{1}{4}\ln\left|4v + 11\right| = 2x + C_1$$

Substitution

$$\begin{aligned} & v = (x - 2y) \\ & \Rightarrow x - 2y - \frac{1}{4}\ln\left|4x - 8y + 11\right| = 2x + C_1 \\ & \Rightarrow 4x - 8y - \ln\left|4x - 8y + 11\right| = 8x + C \\ & \Rightarrow 4x + 8y + \ln\left|4x - 8y + 11\right| = C \end{aligned}$$