Vector differential calculus

Vector algebra

Two kind of physical quantities:

- **Scalar**: mathematical object with 1D property, the **magnitude** : ex. the size or mass, displacement r

- **Vector**: mathematical object that is defined in 3D (or more) - takes into account **degrees of liberty** (dimension) of the action, ex. a displacement also as a **direction** $\vec{r}$; *A vector is a mathematical entity that describes the action possible along different degrees of liberty of a system*

Components of vectors : $\vec{a} = \mathbf{a} = [a_1, a_2, a_3]$:

In Cartesian coordinate system: $$a_1 = x_2 - x_1 \\ a_2 = y_2 - y_1 \\ a_3 = z_2 - z_1$$

Defined as an operation: the components are differences between two values defined within the scalar field that span the vector space ➔ *this is a relative description (ex. = position of object in space)*

Norm (magnitude) of vector: $|\vec{a}| = |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Yields information about the magnitude of the action (weighted by a ***scale*** *or* ***metric****)*

Notion of **(identity) equality** ➔ vector must have same magnitude and direction

To be able to develop operations on vectors we need the following theorem:

Th. 1 Algebraic definition of vector

Given a Cartesian system of coordinates ➔ each vector uniquely determined by its components

Conversely, to each triplet (a_1, a_2, a_3) corresponds precisely one vector $\vec{a}$

Operations

Addition: $\vec{a}+\vec{b}=[a_1+b_1, a_2+b_2, a_3+b_3]$

a) Commutative: $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
b) Associative: $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$

Identity for addition: $\vec{0}=[0,0,0]$

c) $\vec{a}+\vec{0}=\vec{a}$

Inverse for addition: $-\vec{a}=[-a_1,-a_2,-a_2]$

d) $\vec{a}+(-\vec{a})=\vec{0}$

*Geometric equivalent***:** *(Hilbert infinite dimension al space in QM)*
Triangle inequality: *the direction takes some of the weight of action*

e) $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$

Parallelogram equality: *sum on left correspond to two times diagonal – the direction amplified action by factor two*

f) $|\vec{a}+\vec{b}|^2+|\vec{a}-\vec{b}|^2=2\left(|\vec{a}|^2+|\vec{b}|^2\right)$

Scalar multiplication: $c\vec{a}=[ca_1, ca_2, ca_3]$, where c is a scalar (*corresponds to a transformation of scale*)

Basic properties:

a) $c(\vec{a}+\vec{b})=c\vec{a}+c\vec{b}$
b) $(c+k)\vec{a}=c\vec{a}+k\vec{a}$
c) $c(k\vec{a})=(ck)\vec{a}=(kc)\vec{a}=k(c\vec{a})$
d) $1\vec{a}=\vec{a}$

From basic properties of addition and multiplication by scalar:

a) $0\vec{a}=\vec{0}$
b) $(-1)\vec{a}=-\vec{a}$

Identity $|\vec{a}|=1$ and **Normalization**: $\left|\dfrac{\vec{a}}{|\vec{a}|}\right|=1$

Note that there is no division by a vector

Vector space

A **field** is an algebraic structure with notions of ***addition***, ***subtraction***, ***multiplication***, and ***division***, satisfying certain axioms

Any field may be used as the scalars for a vector space

Vector Space: the set of all vectors from $\mathbb{R}^3$ with two algebraic operations, vector addition and scalar multiplication

Unit vectors: $\vec{a} = [a_1, a_2, a_3] = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Where $\hat{i} = [1,0,0]$ $\quad \hat{j} = [0,1,0]$ $\quad \hat{k} = [0,0,1]$, form a **basis**

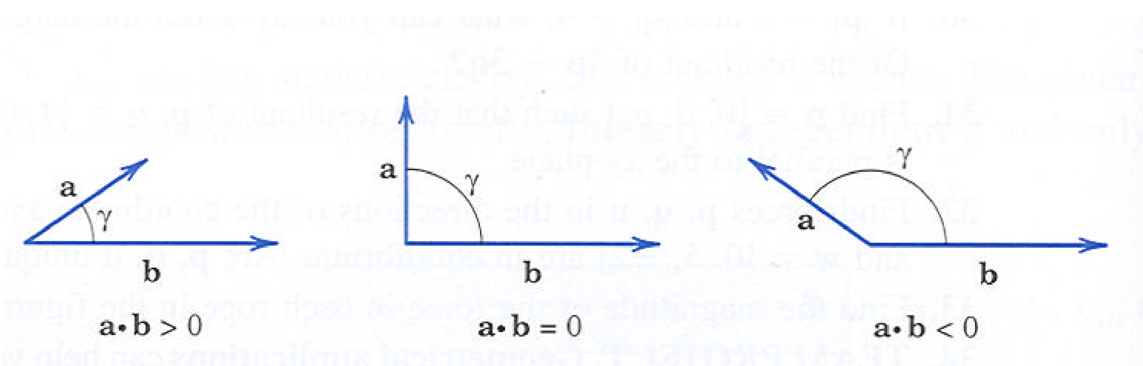
Dimension of $\mathbb{R}^3$ = number of **linearly independent** vector in set – *linearly independent means the actions are independent on these degrees of liberty*

The fact that the vector is defined on different degrees of liberty (directions) introduces 2 new kinds of operations – ***dot product*** *and* ***vector product***

Inner product (dot product)

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\vec{a}||\vec{b}|\cos g \qquad \Rightarrow \cos g = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Where g is the angle between $\vec{a}$ and $\vec{b}$ $\Rightarrow 0 \le g \le p$



The dot product is a ***projection*** *of one vector along the other vector degrees of liberty and yields the resulting action on this degree of liberty – the projection is simultaneously done along the other vector degrees of liberty, because the operation is relative (commutative)* ➔ the result is a **scalar**

For example:

- $\vec{a}\cdot\vec{b} = |\vec{a}|\left(|\vec{b}|\cos g\right)$

  the projection of the action of $\vec{b}$ on $\vec{a}$ degree of liberty

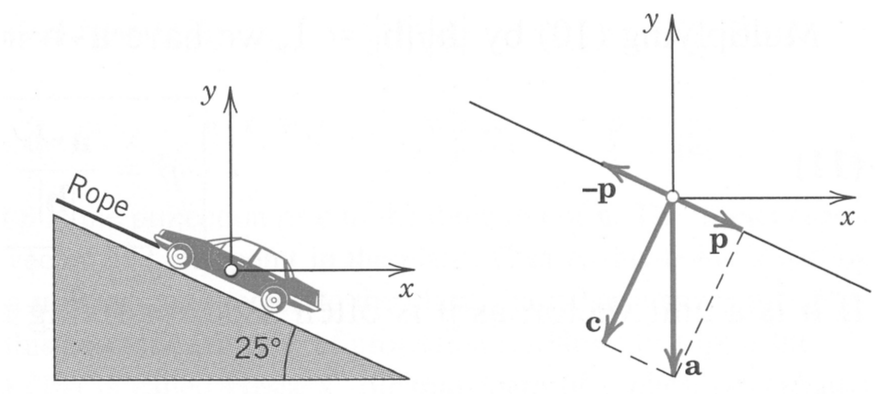
- $\vec{a}\cdot\vec{b} = |\vec{b}|\left(|\vec{a}|\cos g\right)$

  or relatively the projection of the action of $\vec{a}$ on $\vec{b}$ degree of liberty

Ex. 1: Work done by a force

A constant force $\vec{F}$ acts on a body giving it a displacement $\vec{d}$

The **work** done by the force: $W = |\vec{F}||\vec{d}|\cos g = \vec{F}\cdot\vec{d}$; *the work (action) is in the direction of displacement*



To balance the weight of the car the force should be equal to its projection on the ramp:

$$|\vec{p}| = |\vec{a}|\cos g \quad \text{where } g = 90^\circ - 25^\circ = 65^\circ$$

Projection of $\vec{a}$ on $\vec{b}$: $p = |\vec{a}|\cos g$ with $p > 0$ parallel to $\vec{b}$ and $p < 0$ antiparallel to $\vec{b}$

Multiplying by $\frac{|\vec{b}|}{|\vec{b}|} = 1 \Rightarrow p = |\vec{a}|\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\frac{|\vec{b}|}{|\vec{b}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\frac{\vec{b}}{|\vec{b}|}$

In ex. 1, the work done is maximal only if $g = 0$, when the force acts 100% in the direction of the movement

Important case is $g = \frac{p}{2} \Rightarrow \vec{a} \cdot \vec{b} = 0$, *there are no action if the two vectors are along linearly independent degrees of liberty* – the two vectors are **Orthogonal** (linearly independent)

> **Th. 2 Orthogonality**
>
> The inner product of two non zero vectors is zero if and only if they are perpendicular

Consider the two vectors $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\vec{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

In general $\vec{a} \cdot \vec{b} = a_1b_1\mathbf{i} \cdot \mathbf{i} + a_1b_2\mathbf{i} \cdot \mathbf{j} + \cdots + a_3b_3\mathbf{k} \cdot \mathbf{k} = a_1b_1 + a_2b_2 + a_3b_3$

Since $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors, forming a basis of the $\mathbb{R}^3$ space, then:

$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$

There are no interactions between the degrees of liberties $\Rightarrow$ **linearly independent**

Square of inner product = **length of vector** $|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$

The magnitude of a vector is the action of the vector on its own degree of liberty

General properties of inner product

g) **Linearity (distributivity)**: $\left(q_1\vec{a} + q_2\vec{b}\right) \cdot \vec{c} = q_1\vec{a} \cdot \vec{c} + q_2\vec{b} \cdot \vec{c}$

h) **Symmetry (commutativity)**: $\vec{a} \cdot \vec{c} = \vec{c} \cdot \vec{a}$

i) **Positive definiteness**: $\vec{a} \cdot \vec{a} \geq 0 \Rightarrow \vec{a} \cdot \vec{a} = 0 \Rightarrow \vec{a} = \vec{0}$

j) **Schwarz inequality**: $\left|\vec{a} \cdot \vec{b}\right| \leq |\vec{a}|\left|\vec{b}\right|$

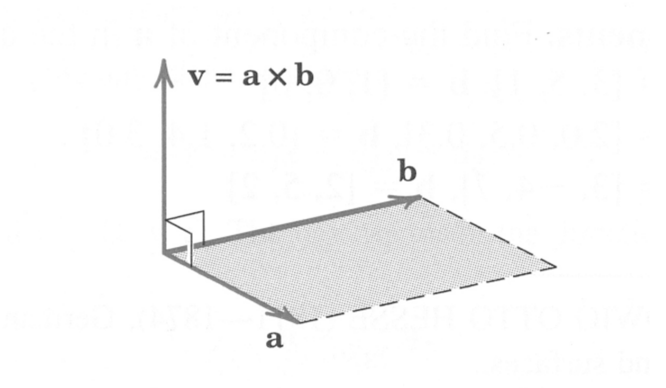
Vector product (cross product)

Defines an interaction between two vectors that has an action along a perpendicular degree of liberties (perpendicular two the plan spanned by the two vectors) $\Rightarrow$ *introduces (or implies) a third dimension* – the result of the cross product is a **vector**

Cross product: $\vec{v} = \vec{a} \times \vec{b}$

$|\vec{v}| = |\vec{a}||\vec{b}|\sin g$ ➔ the area of parallelogram with $\vec{a}$ and $\vec{b}$ as adjacent sides, $\vec{v} \perp$ to $\vec{a}$ and $\vec{b}$

Geometrical interpretation – the parallelogram area spanned by the two vectors can be seen as a vector (note the relation of vector with an area)

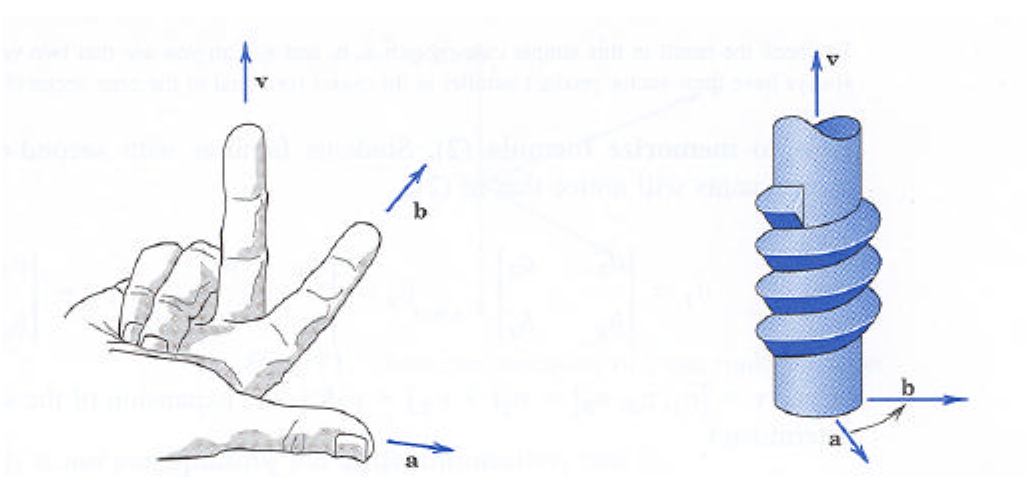


Consequence: if $\vec{a} \,//\, \vec{b} \Rightarrow \vec{v} = \vec{0}$, *no resulting action when the directions are parallel*

NOTE: Ambiguity $\vec{0}$// to any vector but also $\vec{0} \perp$ to any vector

Components: $\vec{v} = [v_1, v_2, v_3]$ (defined arbitrarily) using **right-handed system**

$$v_1 = a_2 b_3 - b_2 a_3 , \qquad v_2 = b_1 a_3 - a_1 b_3 , \qquad v_3 = a_1 b_2 - a_2 b_1$$



Same operation can be deduced using matrix form: taking the **determinant**

$$\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\left(a_2b_3-b_2a_3\right)\hat{i}-\left(a_1b_3-b_1a_3\right)\hat{j}+\left(a_2b_3-b_2a_3\right)\hat{k}$$

Suggest the resulting action depends on the cross interactions between components of degrees of liberty (mixed or cross terms operations)

$$\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\hat{i}-\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}\hat{j}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\hat{k}$$

Ex. 2: Moment of a force (generates movement of rotation)

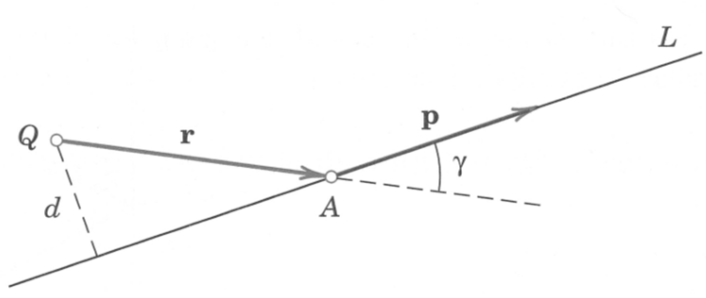
The moment m of a force $\vec{p}$ about a point Q is defined as the product:

$$m=\left|\vec{p}\right|d$$

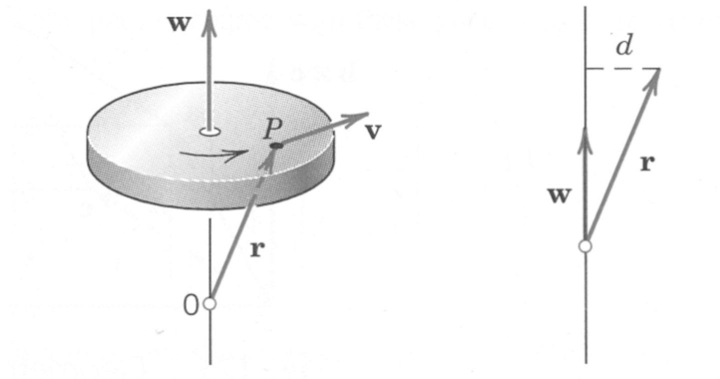
d is the perpendicular distance between Q (**axe of rotation**) and the line of action L of $\vec{p}$; If $\vec{r}$ is the vector from Q to any point A (**point of application of force**) on L , then:

$$d=\left|\vec{r}\right|\sin g \Rightarrow m=\left|\vec{p}\right|\left|\vec{r}\right|\sin g=\left|\vec{r}\times\vec{p}\right|$$

It follows that the moment of force is a vector perpendicular to $\vec{r}$ and $\vec{p}$: $\vec{m}=\vec{r}\times\vec{p}$



Ex. 3: Rotation of a rigid body



Rotation of solid body, uniquely described by $\vec{w}$, **moment of rotation**, which is parallel to axis of rotation

The **angular speed**: $w = |\vec{w}|$

The **tangential speed** $v = wd \Rightarrow |\vec{w}||\vec{r}|\sin g = |\vec{w}\times \vec{r}| \Rightarrow \vec{v} = \vec{w}\times \vec{r}$

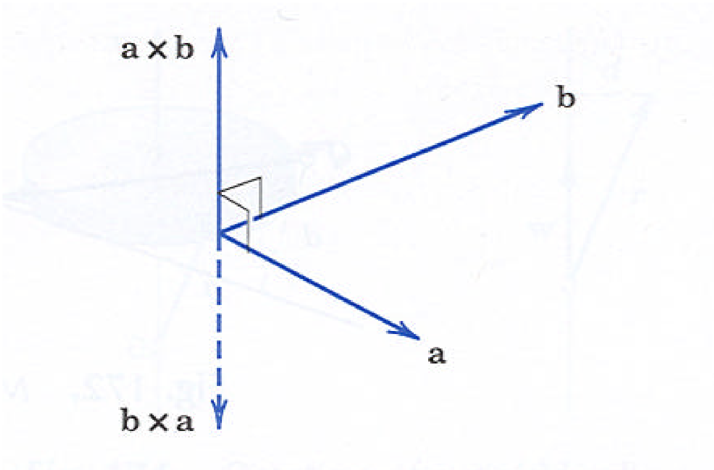
General properties

With l a scalar

a) $(l\vec{a})\times\vec{b} = l\left(\vec{a}\times\vec{b}\right) = \vec{a}\times\left(l\vec{b}\right)$

b) Distributive on addition
$$\vec{a}\times\left(\vec{b}+\vec{c}\right) = \left(\vec{a}\times\vec{b}\right) + \left(\vec{a}\times\vec{c}\right)$$
$$\left(\vec{a}+\vec{b}\right)\times\vec{c} = \left(\vec{a}\times\vec{c}\right) + \left(\vec{b}\times\vec{c}\right)$$

c) Not commutative (symmetric) $\begin{matrix}\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a} \\ \vec{a}\times\vec{b} = -\vec{b}\times\vec{a}\end{matrix}$, *action is inverted (because two inverse directions possible)*



d) Not associative $\left(\vec{a}\times\vec{b}\right)\times\vec{c} \neq \vec{a}\times\left(\vec{b}\times\vec{c}\right)$

Scalar triple product – *generalization of matrix operation*

$$\left(\vec{a} \quad \vec{b} \quad \vec{c}\right) = \vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \vec{a}\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

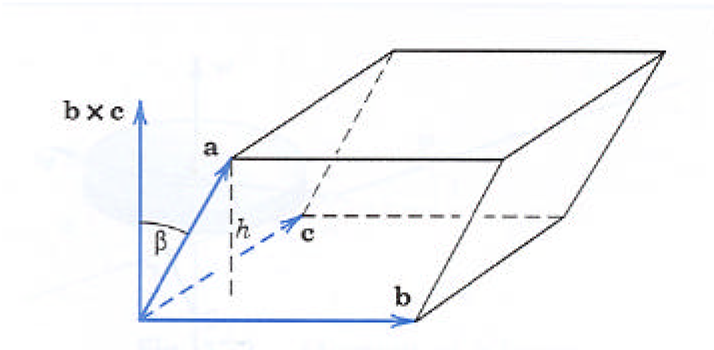
Properties:

a) $\left(k\vec{a} \quad \vec{b} \quad \vec{c}\right) = k\left(\vec{a} \quad \vec{b} \quad \vec{c}\right)$

b) $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$

Geometric interpretation:

$\left|\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right| = \left|\vec{a}\right|\left|\vec{b}\times\vec{c}\right|\cos b$



Since $\left|\vec{b}\times\vec{c}\right|$ is the surface of basis of parallelogram and $\left|\vec{a}\right|\cos b$ is the height of the parallelogram ➔ this is the **volume of parallelogram** with $\vec{a} \quad \vec{b} \quad \vec{c}$ as edge vectors

Th. 1 linear independence of 3 vectors
3 vectors form a linearly independent set if and only if their scalar triple product is not zero

Vector, scalar functions and fields: Derivatives

Scalar function: $f = f(P)$

Where P is a point in space

Scalar functions define a **scalar field**

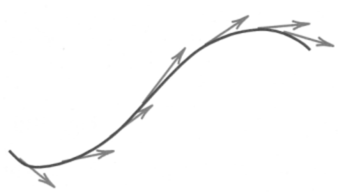
Ex. 1 Euclidean distance = scalar function

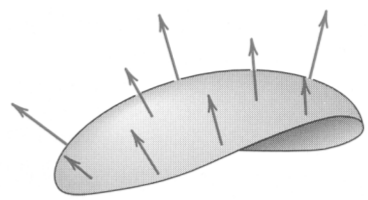
$$f(P) = f(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

Scalar field ➔ invariant in different systems of coordinates

Vector function: $\vec{v} = \vec{v}(P) = [v_1(P), v_2(P), v_3(P)]$

While scalar functions define a **scalar field**, a vector functions defines a **vector field** in a region of space





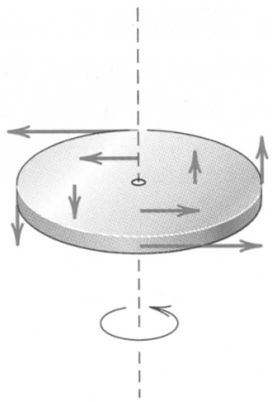
In Cartesian coordinates: $\vec{v}(x, y, z) = [v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)]$

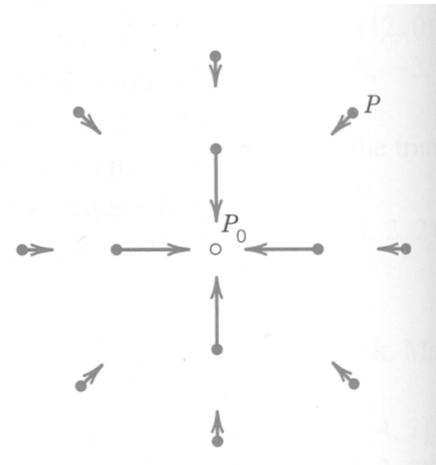
Ex. 2 Velocity field = vector field

Velocity vectors $\vec{v}(P)$ of a rotating body

$$\vec{v}(x,y,z) = \vec{w} \times \vec{r} = \vec{w} \times \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$\Rightarrow \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & w \\ x & y & z \end{vmatrix} = -yw\hat{i} + xw\hat{j} = -w\left(y\hat{i} - x\hat{j}\right)$$





Ex. 3 Force field = vector field

Particle A with mass M at fixed position P_0 and free particle B with mass m at P

Gravitational force: $\left|\vec{F}\right| = \frac{GMm}{r^2} = \frac{C}{r^2}$ and $r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$

Assuming $r > 0 \Rightarrow \vec{r} = (x-x_0)\hat{i} + (y-y_0)\hat{j} + (z-z_0)\hat{k}$

$\left|\vec{r}\right| = r \Rightarrow -\frac{\vec{r}}{r}$ is unit vector in the direction of $\vec{F}$

$$\Rightarrow \vec{F} = \left|\vec{F}\right|\left(-\frac{1}{r}\right)\vec{r} = -\frac{C}{r^3}\vec{r} = -\frac{C}{r^3}(x-x_0)\hat{i} - \frac{C}{r^3}(y-y_0)\hat{j} - \frac{C}{r^3}(z-z_0)\hat{k}$$

Vector calculus

Assumed continuity of action in space (space-time is a continuum)

Convergence: an infinite sequence of vectors $\vec{a}(n)$ $(n=1,2,\ldots)$ converge if there is a vector $\vec{a}$ such that $\lim_{n\to\infty}\left|\vec{a}_n-\vec{a}\right|=0$

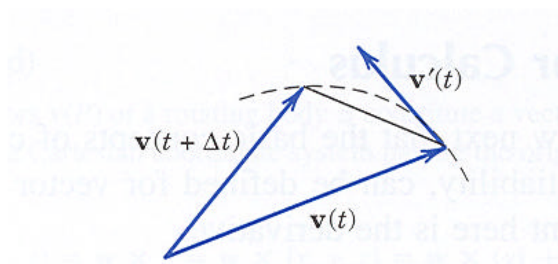
$\vec{a}$ is the limit vector or $\lim_{n\to\infty}\vec{a}_n=\vec{a}$

Similarly, convergence of vector function $\vec{v}(t)$ (defined in neighborhood but not necessarily at t_0) $\lim_{t\to t_0}\left|\vec{v}(t)-\vec{l}\right|=0 \Rightarrow \lim_{t\to t_0}\vec{v}(t)=\vec{l}$

Continuity: $\vec{v}(t)$ continuous at t_0 if defined in neighborhood of t_0 and $\lim_{t\to t_0}\vec{v}(t)=\vec{v}(t_0)$

$\vec{v}(t)=\left[v_1(t),v_2(t),v_3(t)\right]=v_1(t)\hat{i}+v_2(t)\hat{j}+v_3(t)\hat{k}$

Derivative: $\vec{v}'(t)=\lim_{\Delta t\to 0}\dfrac{\vec{v}(t+\Delta t)-\vec{v}(t)}{\Delta t}$



Differentiable only if components are differentiable:

$\vec{v}'(t)=\left[v_1'(t),v_2'(t),v_3'(t)\right]$

Rules of differentiation:

a) $(c\vec{v})'=c\vec{v}'$

b) $(\vec{u}+\vec{v})'=\vec{u}'+\vec{v}'$

c) $(\vec{u}\cdot\vec{v})'=\vec{u}'\cdot\vec{v}+\vec{u}\cdot\vec{v}'$

d) $(\vec{u}\times\vec{v})'=\vec{u}'\times\vec{v}+\vec{u}\times\vec{v}'$

e) $(\vec{u}\quad\vec{v}\quad\vec{w})'=(\vec{u}'\quad\vec{v}\quad\vec{w})+(\vec{u}\quad\vec{v}'\quad\vec{w})+(\vec{u}\quad\vec{v}\quad\vec{w}')$

Ex. 4 derivative of vector function $\vec{v}(t)$ of constant length is either zero or $\perp$ to $\vec{v}(t)$

$\left|\vec{v}\right|^2=\vec{v}\cdot\vec{v}=c^2$ then $(\vec{v}\cdot\vec{v})'=\vec{v}'\cdot\vec{v}+\vec{v}\cdot\vec{v}'=2\vec{v}\cdot\vec{v}'=0$

Partial derivatives

If $\vec{v} = [v_1, v_2, v_3]$ where components are differentiable functions of variables $t_1, t_2, \ldots, t_n$

$$\Rightarrow \frac{\partial \vec{v}}{\partial t_l} = \frac{\partial v_1}{\partial t_l}\hat{i} + \frac{\partial v_2}{\partial t_l}\hat{j} + \frac{\partial v_3}{\partial t_l}\hat{k} \text{ and } \Rightarrow \frac{\partial^2 \vec{v}}{\partial t_l \partial t_m} = \frac{\partial^2 v_1}{\partial t_l \partial t_m}\hat{i} + \frac{\partial^2 v_2}{\partial t_l \partial t_m}\hat{j} + \frac{\partial^2 v_3}{\partial t_l \partial t_m}\hat{k}$$

Differential geometry

Parametric representation: $\vec{r}(t) = [x(t), y(t), z(t)] = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

Other representations:

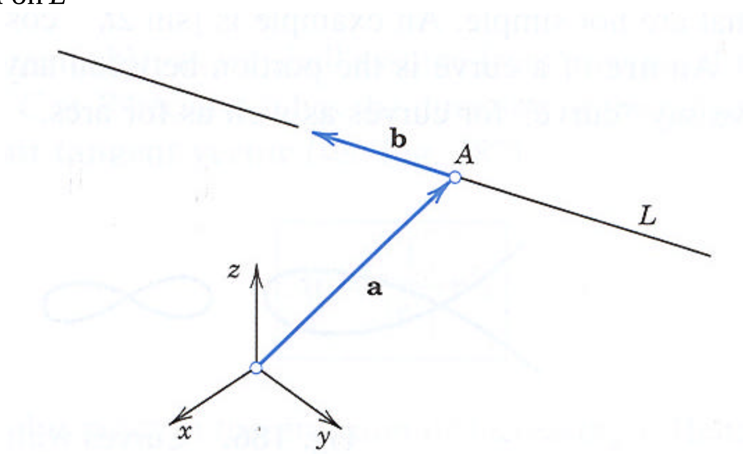
$y = f(x)$ Projection of curve in xy-plane

$z = f(x)$ Projection of curve in xz-plane

Curves as intersection of 2 surfaces: $F(x, y, z) = 0$ and $G(x, y, z) = 0$

Ex. 1 straight line $\vec{r}(t) = \vec{a} + \vec{b}t = [a_1 + b_1 t, a_2 + b_2 t, a_3 + b_3 t]$

If $\vec{b}$ is unit vector, its components are the **direction cosines** of L ➔ $|t|$ measures the distance from A on L



Ex. 2 Ellipse circle (in xy-plane)

$\vec{r}(t) = [x(t), y(t), 0] = [a\cos(t), b\sin(t), 0] = a\cos(t)\hat{i} + b\sin(t)\hat{j}$

Principal axes: x and y direction

Norm: $|\vec{r}(t)|^2 = x^2 + y^2$, and since $x = a\cos t$ and $y = b\sin t$

$\Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ if $a = b$ ➔ circle

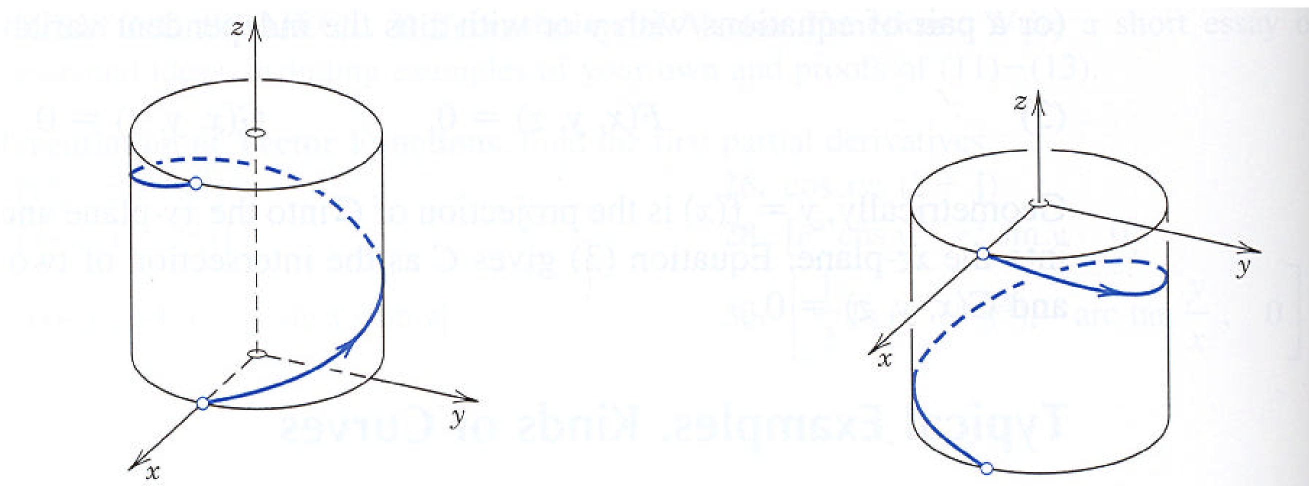
Ex. 3 Circular helix

Twisted curve represented by the vector function:

$\vec{r}(t) = [a\cos t, a\sin t, ct] = a\cos t\mathbf{i} + a\sin t\mathbf{j} + ct\mathbf{k}$, with $c \neq 0$

It lies on the cylinder: $x^2 + y^2 = a^2$

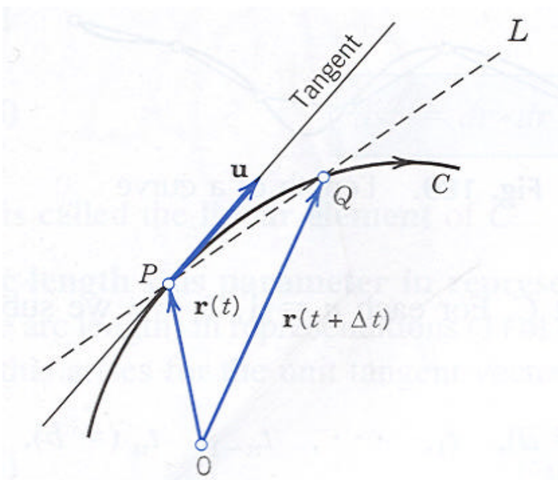
For $c > 0 \Rightarrow$ right handed screw (left otherwise). If $c = 0$ then it is a circle



Tangent to a curve C at point P is the limiting position of straight line L through P and a point Q of C as Q approaches P along C

Following vector has the direction of L : $\dfrac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}$

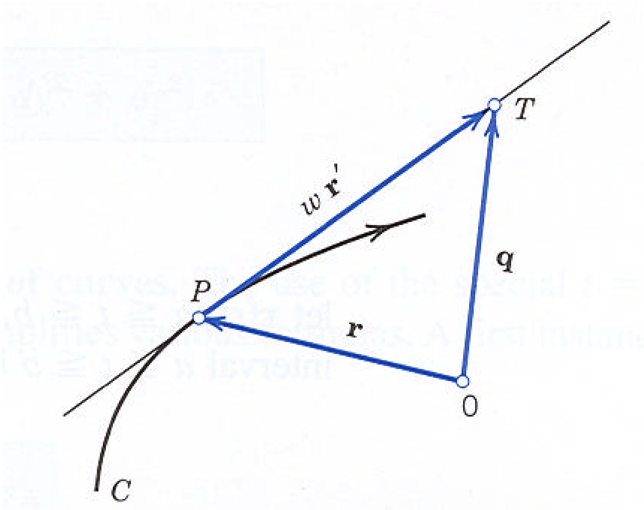
At the limit $\Delta t \to 0$ ➔ $\vec{r}'(t)=\lim_{\Delta t\to 0}\dfrac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}$



Tangent vector: $\vec{r}'(t)$

Unit tangent vector: $\hat{u}=\dfrac{\vec{r}'(t)}{\left|\vec{r}'(t)\right|}$

Tangent to C **at point** $P:\vec{q}(w)=\vec{r}+w\vec{r}'$



Ex. 4 Tangent to an ellipse

We search the tangent to the ellipse $\frac{1}{4}x^2+y^2=1$

at $P:\left(\sqrt{2},1/\sqrt{2}\right)$, which correspond to $t=p/4$

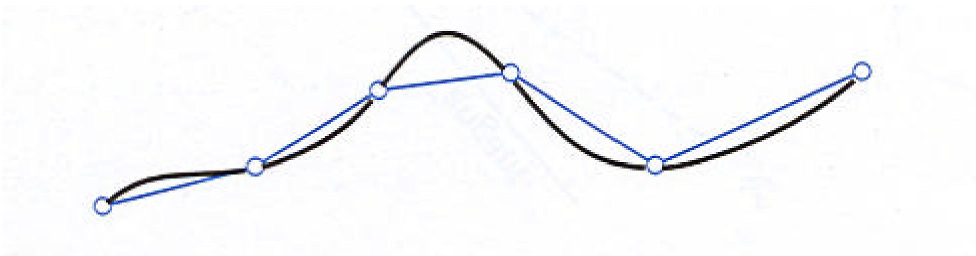
Here we have $\vec{r}(t)=2\cos t\mathbf{i}+\sin t\mathbf{j}$

$\Rightarrow \vec{r}'(t)=-2\sin t\mathbf{i}+\cos t\mathbf{j}$

Since $2\cos(p/4)=\sqrt{2}$ and $\sin(p/4)=1/\sqrt{2}$ then $\vec{r}'\left(\frac{p}{4}\right)=-\sqrt{2}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}$:

$\mathbf{q}(w)=\left[\sqrt{2},1/\sqrt{2}\right]+w\left[-\sqrt{2},1/\sqrt{2}\right]=\sqrt{2}(1-w)\mathbf{i}+1/\sqrt{2}(1+w)\mathbf{j}$

Arc length



Let $\vec{r}(t),\ a \le t \le b$ represent C

We partition the interval by n points: $t_0 (= a),\ t_1,\ \ldots,\ t_{n-1},\ t_n (= b)$

We do that arbitrarily so that the greatest $\lim_{n\to\infty} |\Delta t_m| = |t_m - t_{m-1}| = 0$

The lengths $l_1,\ l_2,\ \ldots$ can be obtained by Pythagorean theorem

If $\vec{r}(t)$ has continuous derivative $\vec{r}'(t) = d\vec{r}/dt$ the sequence $l_1,\ l_2,\ \ldots$ has a limit, which is independent of the representation of C and of the choice of the subdivisions.

The limit is given by the integral:

Length of curve: $l = \int_a^b \sqrt{\vec{r}' \cdot \vec{r}'}\,dt$ ➔ C is rectifiable

Arc length s **of curve** C: $s(t) = \int_0^t \sqrt{\vec{r}' \cdot \vec{r}'}\,d\tilde{t}$

Linear element ds : $\left(\frac{ds}{dt}\right)^2 = \frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt} = |\vec{r}'(t)|^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$

$\Rightarrow d\vec{r} = (dx, dy, dz) = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$\Rightarrow ds^2 = dx^2 + dy^2 + dz^2$

Using Arc Length as parameter to a curve ➔ simplification of formulae

For example, the **unit tangent vector**: $\hat{u} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \Rightarrow \vec{u}(s) = \vec{r}'(s)$

Ex. 5 Circular Helix

$\vec{r}(t) = [a\cos t, a\sin t, ct] = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\,\mathbf{k}$

The derivative:

$$\vec{r}'(t) = \frac{d\vec{r}}{dt} = [-a\sin t, a\cos t, c] = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + c\,\mathbf{k}$$

From this we get: $\mathbf{r}' \cdot \mathbf{r}' = a^2 + c^2$

So that $s = \int_0^t \sqrt{a^2 + c^2}\, d\tilde{t} = t\sqrt{a^2 + c^2}$

Hence $t = \dfrac{s}{\sqrt{a^2 + c^2}}$ and a formula for the helix

$$\vec{r}^*(s) = \vec{r}^*\left(\frac{s}{\sqrt{a^2 + c^2}}\right) = a\cos\frac{s}{\sqrt{a^2 + c^2}}\mathbf{i} + a\sin\frac{s}{\sqrt{a^2 + c^2}}\mathbf{j} + \frac{cs}{\sqrt{a^2 + c^2}}\mathbf{k}$$

For a circle $c = 0$ in the counter-clock wise direction

$$\vec{r}\left(\frac{s}{a}\right) = a\cos\frac{s}{a}\mathbf{i} + a\sin\frac{s}{a}\mathbf{j}$$

Setting $s = -s^*$ and using $\cos(-a) = \cos a$ and $\sin(-a) = -\sin a$

$$\vec{r}\left(-\frac{s^*}{a}\right) = a\cos\frac{s^*}{a}\mathbf{i} - a\sin\frac{s^*}{a}\mathbf{j}$$

For a circle in the clock-wise direction

Curves in mechanics: velocity and acceleration

Parameter t = **time**

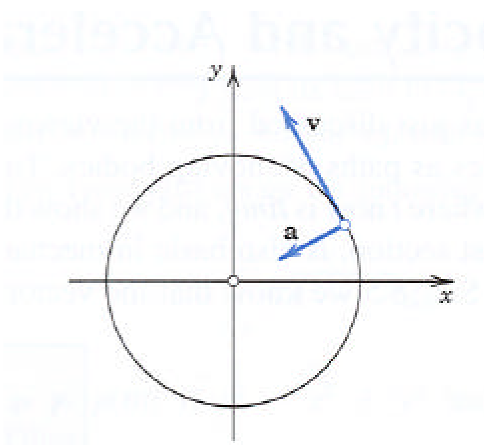
Velocity: $\vec{v} = \vec{r}' = \dfrac{d\vec{r}}{dt}$ or $\mathbf{v} = \dot{\mathbf{r}}$

Speed (magnitude of velocity): $|\mathbf{v}| = \sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}} = \dfrac{ds}{dt} = v$

Acceleration: $\dot{\mathbf{v}} = \mathbf{a}(t) = \dfrac{d^2\vec{r}}{dt^2}$

Ex. 1 Centripetal acceleration

Circle of radius R with center at origin in xy-plane: $\mathbf{r}(t) = R\cos wt\,\mathbf{i} + R\sin wt\,\mathbf{j}$



Rotation in counter clock wise sense w = **angular speed**

$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = -Rw\sin wt\,\mathbf{i} + Rw\cos wt\,\mathbf{j}$

Magnitude = **rotation speed**: $|\mathbf{v}| = Rw$ ➔ **angular speed**: $w = \dfrac{|\mathbf{v}|}{R}$

Acceleration: $\dot{\mathbf{v}}(t) = -Rw^2\cos wt\,\mathbf{i} - Rw^2\sin wt\,\mathbf{j} = -w^2\mathbf{r}$ ➔ points towards the origin

Magnitude = **centripetal acceleration**: $|\mathbf{a}| = w^2 R$

Tangential and normal acceleration

Applying chain rule for differentiation to curves:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{u}(s)\frac{ds}{dt}$$ where $\mathbf{u}(s)$ is **unit tangent vector** of C

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{u}(s)\frac{ds}{dt}\right) = \frac{d\mathbf{u}}{ds}\left(\frac{ds}{dt}\right)^2 + \mathbf{u}(s)\frac{d^2 s}{dt^2}$$

Since $\mathbf{u}(s)$ is tangent to C and of constant length and $\frac{d\mathbf{u}}{ds} \perp$ to $\mathbf{u}(s)$

Normal acceleration: $\frac{d\mathbf{u}}{ds}\left(\frac{ds}{dt}\right)^2 = \frac{d\mathbf{u}}{ds}v^2$

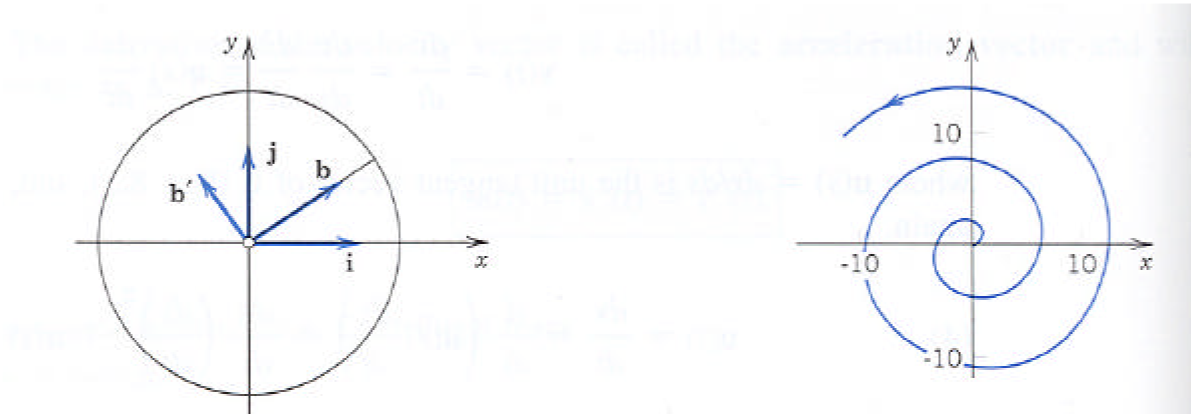
Tangential acceleration: $\mathbf{u}(s)\frac{d^2 s}{dt^2} = \mathbf{u}(s)\frac{dv}{dt}$

$\Rightarrow \mathbf{a} = \mathbf{a}_{norm} + \mathbf{a}_{\tan}$

The length of $\mathbf{a}_{\tan}$ is the projection of $\mathbf{a}$ in the direction of $\mathbf{v}$

Since $|\mathbf{a}_{\tan}| = \frac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|} \Rightarrow \mathbf{a}_{\tan} = \frac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|}\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}$

Ex. 2 Coriolis acceleration



Small body B with mass m_0 moves toward the edge of a rotating disk rotating with angular rotation speed: $w=1$

Position vector: $\mathbf{r}(t)=t\mathbf{b}$ where $\mathbf{b}=$ unit vector

Rotating with disk ➔ $\mathbf{b}=\cos wt\,\mathbf{i}+\sin wt\,\mathbf{j}=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}$

Velocity: $\mathbf{v}=\dfrac{d\mathbf{r}}{dt}=\mathbf{b}+t\dot{\mathbf{b}}$

- where $\mathbf{b}$ is the velocity of B relative to the disk
- and $t\dot{\mathbf{b}}$ is an additional velocity due to the rotation

Acceleration: $\mathbf{a}=\dot{\mathbf{v}}=2\dot{\mathbf{b}}+t\ddot{\mathbf{b}}$ but since $\ddot{\mathbf{b}}=-\mathbf{b}\Rightarrow t\ddot{\mathbf{b}}=-t\mathbf{b}=$ **centripetal acceleration**

Coriolis acceleration: $2\dot{\mathbf{b}}$ in the direction $\perp$ to $\mathbf{b}$ tangential to the edge of the disk

The body B feels a force $-2m_0\dot{\mathbf{b}}$ against the sense of rotation (reaction due to friction)

In the fixed $xy-$plane, B describes a spiraling ellipse

Tangential acceleration: $\mathbf{a}_{\tan}=\dfrac{\mathbf{a}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}=\dfrac{(2\dot{\mathbf{b}}+t\ddot{\mathbf{b}})\cdot(\mathbf{b}+t\dot{\mathbf{b}})}{|\mathbf{b}+t\dot{\mathbf{b}}|^2}(\mathbf{b}+t\dot{\mathbf{b}})=\dfrac{t}{1+t^2}(\mathbf{b}+t\dot{\mathbf{b}})$

Normal acceleration: $\mathbf{a}_{norm}=\mathbf{a}-\mathbf{a}_{\tan}=\dfrac{2+t^2}{1+t^2}(\dot{\mathbf{b}}-t\mathbf{b})$

Ex. 3 superposition of 2 rotations

On Earth, uniform motion of projectile B along a meridian M

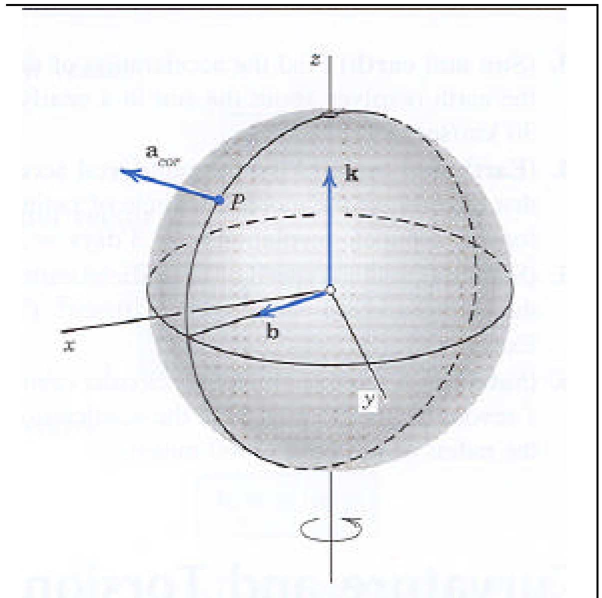
$$\mathbf{r}(t) = R\cos g t\,\mathbf{b} + R\sin g t\,\mathbf{k}$$

where

$$\mathbf{b}(t) = \cos wt\,\mathbf{i} + \sin wt\,\mathbf{j}$$

$w > 0$ is the angular speed of Earth

g = angular speed of the projectile



$$\dot{\mathbf{r}}(t) = R\cos g t\,\dot{\mathbf{b}} - g R\sin g t\,\mathbf{b} + g R\cos g t\,\mathbf{k}$$

$$\ddot{\mathbf{r}}(t) = R\cos g t\,\ddot{\mathbf{b}} - 2g R\sin g t\,\dot{\mathbf{b}} - g^2 R\cos g t\,\mathbf{b} - g^2 R\sin g t\,\mathbf{k}$$

Centripetal acceleration due to rotation of the Earth: $\mathbf{a}_{cent} = R\cos g t\,\ddot{\mathbf{b}}$

Coriolis acceleration: $\mathbf{a}_{cor} = -2g R\sin g t\,\dot{\mathbf{b}}$

Centripetal acceleration due to the motion of B **on** M : $-g^2\left[R\cos g t\,\mathbf{b} + R\sin g t\,\mathbf{k}\right]$

In the northern hemisphere, $\sin g t > 0$ ➔ $\mathbf{a}_{cor}$ points in direction $-\dot{\mathbf{b}}$, opposite to direction of the rotation of the Earth

$|\mathbf{a}_{cor}|$ is maximum at the pole and zero at the equator

The effect of Coriolis force ma_{cor} makes B deviates from M to the right (to the left in southern hemisphere)

Historically, the concept of function developed in terms of curves arising from geometrical or mechanical problems

Some of them became famous as object of extensive studies

Here are some of them:

Archimedes spiral	$r = a\theta$
Hyperbolic spiral	$r = a/\theta,$
Logarithmic spiral	$r = ae^{b\theta}$
Astroid (four-cusped hypocycloid)	$x = 4\cos^3 t,\ y = 4\sin^3 t$
Cardioid	$r = a(1 - \cos\theta)$
Descartes folium	$r = \dfrac{3a\sin 2\theta}{\cos^3\theta + \sin^3\theta}$
Diocles cissoid	$r = \dfrac{2a\sin^2\theta}{\cos\theta}$
Lamé curve	$x^4 + y^4 = 1$
Maclaurin trisectrix	$r = 2a\,\dfrac{\sin 3\theta}{\sin 2\theta}$
Nicomedes conchoid	$r = \dfrac{a}{\cos\theta} + b$
Pascal limaçon (*snail*)	$r = 2a\cos\theta + b$
Steiner hypocycloid	$x = 2\cos t + \cos 2t, \quad y = 2\sin t - \sin 2t$

Curvature

Curvature of curve C represented by $\mathbf{r}(s)$ with arc length s:

$$k(s) = \left|\frac{d\mathbf{u}}{ds}\right| = |\mathbf{u}'(s)| = \left|\frac{d^2\mathbf{r}}{ds^2}\right| = |\mathbf{r}''(s)|$$

Where $\mathbf{u}(s) = \mathbf{r}'(s)$ = **unit tangent vector** of C (we assume $\mathbf{r}(s)$ twice differentiable)

*The **curvature** is the length of the rate of change of the unit tangent vector with s*

➔ Measures deviation of C from the tangent

For general parameter t ➔ formula for curvature is complicated

Other definitions using curvature:

Principal normal vector: $\mathbf{p}(s) = \frac{1}{|\mathbf{u}'|}\mathbf{u}' = \frac{1}{k}\mathbf{u}'$

Unit binormal vector: $\mathbf{b}(s) = \mathbf{u} \times \mathbf{p}$

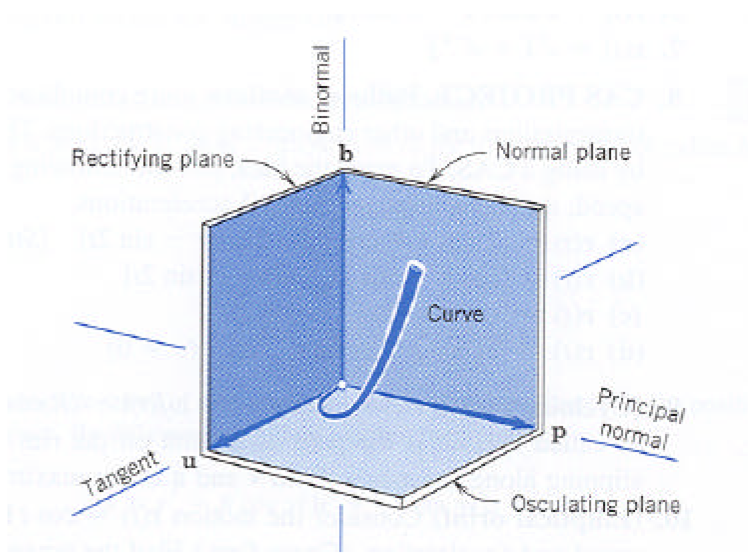
Trihedron of curve C

Straight line in direction of tangent $\mathbf{u}$, binormal $\mathbf{b}$ and principal normal $\mathbf{p}$

Rectifying plane:
$\mathbf{ub}$ – *plane*

Osculating plane:
$\mathbf{up}$ – *plane*

Normal plane:
$\mathbf{bp}$ – *plane*



Torsion of a curve

Measures the deviation of C from the osculating plane

Torsion of curve C: $t(s) = -\mathbf{p}(s)\cdot\mathbf{b}'(s)$

Negative sign makes torsion positive for right-handed helix

For plane curve ➔ $\mathbf{b}$ is constant ➔ $t(s) = 0$

If $\mathbf{b}'$ exist its direction is along $\mathbf{p}$ ➔ $\mathbf{b}' \perp$ to $\mathbf{b}$ and $\mathbf{u}$

$\Rightarrow \mathbf{b}' = -t\,\mathbf{p}$

Since $\mathbf{u}$, $\mathbf{b}$ and $\mathbf{p}$ linearly independent vectors ➔ form basis ➔ any vector could be represented as a linear combination of this basis

Frenet's Formula:
$$\mathbf{u}' = k\mathbf{p}$$
$$\mathbf{p}' = -k\mathbf{u} + t\,\mathbf{b}$$
$$\mathbf{b}' = -t\,\mathbf{p}$$

A curve is uniquely determined (except for position in space) if we prescribe its curvature $k > 0$ and torsion t as continuous function of arc length s

$k = k(s)$ and $t = t(s)$ **natural equations of a curve**

Ex. 1 Circular helix

For a circular helix: $s = t\sqrt{a^2 + c^2}$

Hence $\mathbf{r}(s) = a\cos\frac{s}{K}\mathbf{i} + a\sin\frac{s}{K}\mathbf{j} + c\frac{s}{K}\mathbf{k}$ where $K = \sqrt{a^2 + c^2}$

It follows that: $\mathbf{u}(s) = \mathbf{r}'(s) = -\frac{a}{K}\sin\frac{s}{K}\mathbf{i} + \frac{a}{K}\cos\frac{s}{K}\mathbf{j} + \frac{c}{K}\mathbf{k}$

$$\mathbf{r}''(s) = -\frac{a}{K^2}\cos\frac{s}{K}\mathbf{i} - \frac{a}{K^2}\sin\frac{s}{K}\mathbf{j}$$

$$k(s) = |\mathbf{r}''| = \sqrt{\mathbf{r}''\cdot\mathbf{r}''} = \frac{a}{K^2} = \frac{a}{a^2 + c^2}$$

$$\mathbf{p}(s) = \frac{1}{k(s)}\mathbf{r}''(s) = -\cos\frac{s}{K}\mathbf{i} - \sin\frac{s}{K}\mathbf{j}$$

$$\mathbf{b}(s) = \mathbf{u}(s) \times \mathbf{p}(s) = \frac{c}{K}\sin\frac{s}{K}\mathbf{i} - \frac{c}{K}\cos\frac{s}{K}\mathbf{j} + \frac{a}{K}\mathbf{k}$$

$$\mathbf{b}'(s) = \frac{c}{K^2}\cos\frac{s}{K}\mathbf{i} + \frac{c}{K^2}\sin\frac{s}{K}\mathbf{j}$$

$$t(s) = -\mathbf{p}(s)\cdot\mathbf{b}'(s) = \frac{c}{K^2} = \frac{c}{a^2 + c^2}$$

Hence, a circular helix has constant curvature and torsion

For a circle $c = 0$ and $k = \frac{1}{a}$ the reciprocal of the radius, while $t = 0$; also $\mathbf{b}(s) = \mathbf{k}$ is constant perpendicular to the plane of the circle

Calculus in several variables

Partial derivatives

If $\vec{v} = [v_1, v_2, v_3]$ where components are differentiable functions of variables $t_1, t_2, \ldots, t_n$

$$\Rightarrow \frac{\partial \vec{v}}{\partial t_l} = \frac{\partial v_1}{\partial t_l}\hat{i} + \frac{\partial v_2}{\partial t_l}\hat{j} + \frac{\partial v_3}{\partial t_l}\hat{k} \text{ and } \Rightarrow \frac{\partial^2 \vec{v}}{\partial t_l \partial t_m} = \frac{\partial^2 v_1}{\partial t_l \partial t_m}\hat{i} + \frac{\partial^2 v_2}{\partial t_l \partial t_m}\hat{j} + \frac{\partial^2 v_3}{\partial t_l \partial t_m}\hat{k}$$

Th. 1 Chain rule

Let $w = f(x, y, z)$ be continuous + first derivatives in domain D in xyz-space and $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ be continuous function + first derivatives in domain B in uv-plane, where B is such that for every point (u,v) in B corresponding point $(x(u,v), y(u,v), z(u,v))$ lies in D

$\Rightarrow w = f(x(u,v), y(u,v), z(u,v))$ defined in B and

$$\Rightarrow \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\Rightarrow \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

Of particular interest: $x = x(t)$, $y = y(t)$, $z = z(t)$

$$\Rightarrow \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Ex. 1 $w = x^2 - y^2$ where $x = r\cos q$ and $y = r\sin q$

$$\Rightarrow \frac{dx}{dr} = \cos q \text{ and } \frac{dx}{dq} = -r\sin q$$

$$\Rightarrow \frac{dy}{dr} = \sin q \text{ and } \frac{dx}{dq} = r\cos q$$

$$\frac{\partial w}{\partial r} = 2x\cos q - 2y\sin q = 2r\cos^2 q - 2r\sin^2 q = 2r\cos 2q$$

$$\frac{\partial w}{\partial q} = -2xr\sin q - 2yr\cos q = -2r^2\cos q\sin q - 2r^2\cos q\sin q = -2r^2\sin 2q$$

Th. 2 Mean value theorem

Consider $f(x, y, z)$ continuous with continuous derivatives in domain D

And let $P_0 : (x_0, y_0, z_0)$ and $P : (x_0 + h, y_0 + k, z_0 + l)$ points in D such that the straight line through these points lies entirely within D then

$$f(x_0 + h, y_0 + k, z_0 + l) - f(x_0, y_0, z_0) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z}$$

Gradient of scalar field

Some (not all) vector fields can be obtained from scalar field

➔ Particularly important in physics

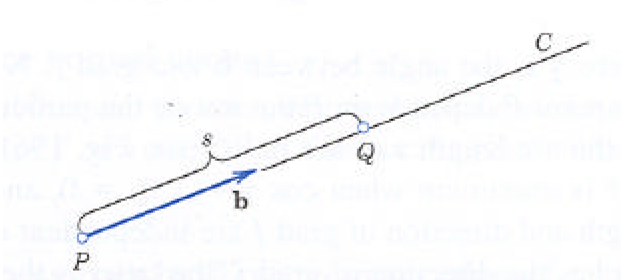
Gradient

Nabla = **differential operator** $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

Gradient: $\text{grad}\, f = \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

Vector with length and direction independent of coordinate system

Directional derivative



The rate of change of f at any point P in any fixed direction given by $\mathbf{b}$

➔ **directional derivative**: $D_{\mathbf{b}} f = \frac{df}{ds} = \lim_{s \to 0} \frac{f(Q) - f(P)}{s}$

Where s is the distance between points P and Q

Q = variable point on ray C in direction $\mathbf{b}$

Using Cartesian coordinates and $|\mathbf{b}| = 1$

$$\mathbf{r}(s) = x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k} = \mathbf{p}_0 + s\mathbf{b}$$

Applying chain rule: $D_{\mathbf{b}} f = \dfrac{df}{ds} = \dfrac{\partial f}{\partial x} x' + \dfrac{\partial f}{\partial y} y' + \dfrac{\partial f}{\partial z} z'$

However, since $\mathbf{r}' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k} = \mathbf{b} \Rightarrow D_{\mathbf{b}} f = \dfrac{df}{ds} = \mathbf{b} \cdot \vec{\nabla} f$

For vector $\mathbf{a}$ of any length: $\Rightarrow D_{\mathbf{a}} f = \dfrac{df}{ds} = \dfrac{1}{|\mathbf{a}|} \mathbf{a} \cdot \vec{\nabla} f$

Ex. 1

We search for the directional derivative of $f(x, y, z) = 2x^2 + 3y^2 + z^2$

at the point $P:(2,1,3)$ in the direction of the vector $\mathbf{a} = \mathbf{i} - 2\mathbf{k}$

$\vec{\nabla} f = 4x\mathbf{i} + 6y\mathbf{j} + 2z\mathbf{k}$ and at $P \Rightarrow \vec{\nabla} f = 8\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$

$$D_a f = \frac{1}{\sqrt{5}} (\mathbf{i} - \mathbf{2k}) \cdot (8\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}) = -\frac{4}{\sqrt{5}}$$

The minus sign indicates that f decreases at P in the direction of $\mathbf{a}$

Gradient characterize maximum increase

Th. 1 Gradient and maximum increase

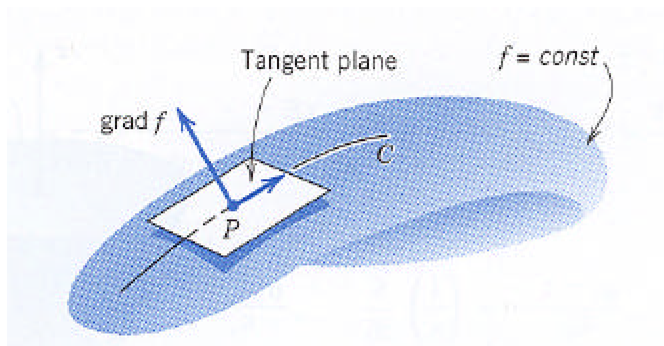
Let $f(P) = f(x, y, z)$ a scalar function having continuous first partial derivatives

➔ $\vec{\nabla} f$ exists and its length and direction are independent of any particular choice of Cartesian coordinates system

If at P, $\vec{\nabla} f \neq \vec{0}$ its direction indicates direction of maximum increase of f

Gradient as surface normal vector

Consider a surface $f(x, y, z) = c$



Curve C in space:
$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

For C to lies on surface

➔ $f(x(t), y(t), z(t)) = c$

Tangent vector of C: $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$

If C lies on S ➔ $\mathbf{r}'$ tangent to S at P

The set of tangent vectors at P form **tangent plane**

Normal to this plane = **surface normal** and surface normal vectors // to this surface

Differentiating f on the surface: $\frac{\partial f}{\partial x} x' + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial z} z' = 0$

That is $\vec{\nabla} f \cdot \mathbf{r}' = 0 \Rightarrow \vec{\nabla} f \perp \mathbf{r}'$

Th. 2 Gradient and surface normal

If f = differentiable scalar function representing a surface S and if $\vec{\nabla} f \neq \mathbf{0}$ at P, then $\vec{\nabla} f$ is normal vector of S at P

Potential

Vector fields that are gradient of a scalar field = **Conservative potential**

Potential function $f(P)$ such that $\mathbf{V}(P) = \vec{\nabla} f(P)$ ➔ **vector field**

In such vector fields, energy is conserved ➔ no energy lost or gained in displacing a body from point P to another point into the field and back to P

Ex. 1 gravitational field (Laplace's equation)

Force of attraction between 2 particles

$$\mathbf{F} = -\frac{C}{r^3}\mathbf{r} = -C\left(\frac{x-x_0}{r^3}\mathbf{i} + \frac{y-y_0}{r^3}\mathbf{j} + \frac{z-z_0}{r^3}\mathbf{k}\right)$$

Where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$

Note that $\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{(x-x_0)}{r^3}$, $\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{(y-y_0)}{r^3}$, and $\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{(z-z_0)}{r^3}$

$$\Rightarrow \mathbf{F} = \vec{\nabla}\left(\frac{C}{r}\right) = \vec{\nabla} f$$

where $f = \frac{C}{r}$ is the potential of the gravitational field

(or any potential defined, differing by a constant: $f(P) + k$)

➔ $f(P)$ satisfies **Laplace's equation**: $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$

Laplacian operator: $\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

The field of force produced by any distribution of masses is given by a vector function that is the gradient of a scalar function f satisfying $\Delta f = 0$ in **any region of space free of matter**

Divergence

Let $\mathbf{v}(x, y, z)$ be a differentiable vector function of Cartesian coordinates and v_1, v_2, v_3 the components of $\mathbf{v}$, then the **divergence**:

$$div\ \mathbf{v} = \vec{\nabla} \cdot \mathbf{v} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \cdot \left(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \right) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Th. 1 Invariance of the divergence

Values of $\vec{\nabla} \cdot \mathbf{v}$ depend only on points in space ➔ independent of coordinates

Laplacian = divergence of gradient: $\vec{\nabla} \cdot \vec{\nabla} f = \nabla^2 f$

Ex. 1 Gravitational force

$$\mathbf{F} = \vec{\nabla} f = \vec{\nabla}\left(\frac{C}{r} \right)$$

Laplace's equation: $\nabla^2 f = 0$ ➔ $\vec{\nabla} \cdot \mathbf{F} = 0$

Ex. 2 Continuity equation of compressible fluid

Motion of fluid in region R without sources or sinks ➔ **compressible fluid** $r = r(x, y, z)$

Flow through rectangular box W of dimension $\Delta x, \Delta y, \Delta z$ ➔ Volume: $\Delta V = \Delta x \Delta y \Delta z$

Velocity vector: $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

Change of mass inside W : $\boldsymbol{\mu} = r\,\mathbf{v} = [m_1, m_2, m_3] = m_1\,\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k}$

Flux across boundaries:

Surface normal to $y : \Delta x \Delta z$ ➔ entering $(r v_2)_y \, \Delta x \Delta z \Delta t$
leaving $(r v_2)_{y+\Delta y} \, \Delta x \Delta z \Delta t$

Difference: $\dfrac{(r v_2)_{y+\Delta y} - (r v_2)_y}{\Delta y} \Delta V \Delta t = \dfrac{\Delta u_2}{\Delta y} \Delta V \Delta t$

Considering all sides: $\left(\dfrac{\Delta u_1}{\Delta x} + \dfrac{\Delta u_2}{\Delta y} + \dfrac{\Delta u_3}{\Delta z} \right) \Delta V \Delta t$

Change of mass = time rate of change of density: $-\dfrac{\partial r}{\partial t} \Delta V \Delta t$

If we let $\Delta x, \Delta y, \Delta z, \Delta t \rightarrow 0$ then $\vec{\nabla} \cdot \vec{m} = \vec{\nabla} \cdot r\vec{v} = -\dfrac{\partial r}{\partial t}$

Continuity equation: $\vec{\nabla} \cdot r\vec{v} + \dfrac{\partial r}{\partial t} = 0$

For a steady flow: $\dfrac{\partial r}{\partial t} = 0 \Rightarrow \vec{\nabla} \cdot r\,\vec{v} = 0$ for $r =$ constant $\Rightarrow \vec{\nabla} \cdot \vec{v} = 0$ **this is the condition of incompressibility**

Curl of Vector

Consider the differentiable vector function $\mathbf{v}(x,y,z) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

Curl (or rotational): $\vec{\nabla}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$

Ex. 1 Rotation of a rigid body

Angular speed: $|\mathbf{?}| = w$

Velocity field of rotation: $\mathbf{v} = \mathbf{?} \times \mathbf{r}$ where $\mathbf{r}$ = position of rotating point

For right-handed coordinates: $\mathbf{?} = w\mathbf{k}$ and $\mathbf{v} = \mathbf{?}\times\mathbf{r} = -wy\mathbf{i} + wx\mathbf{j}$

$$\vec{\nabla}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -wy & wx & 0 \end{vmatrix} = 2w\mathbf{k} = 2\mathbf{?}$$

Important properties:

1. $\vec{\nabla}\times\vec{\nabla}f = 0$

   Gradient fields describing a motion are **irrotational ➔ field is conservative**

2. $\vec{\nabla}\cdot\left(\vec{\nabla}\times\mathbf{v}\right) = 0$

Th. 1 invariance of the curl

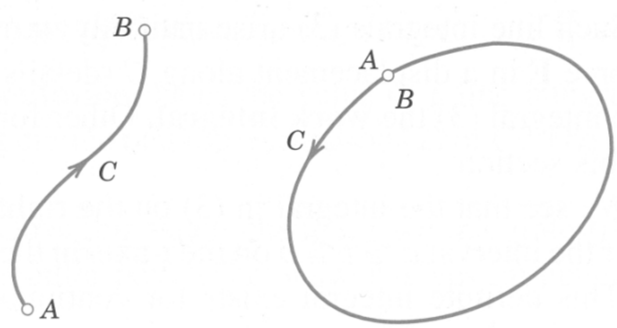
The length and direction of the curl of a vector are independent of the coordinate system

Line integral

Generalization of **definite integral**: $\int_a^b f(x)\,dx$

Consider the curve C: $\mathbf{r}(t)=[x(t),y(t),z(t)]=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$

C = path of integration with $\mathbf{r}(a)$ = initial point, $\mathbf{r}(b)$ = final point and t increases in positive direction



(a) (b)

Smooth curve: C has unique tangential at each points whose direction varies continuously as we move along C

$\Rightarrow \mathbf{r}(t)$ differentiable and $\mathbf{r}'$ continuous different from zero

Piecewise smooth: every integration path consists of finitely many smooth curves

Line integral of vector $\mathbf{F}(\mathbf{r})$ over curve C: $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}=\int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt$

In terms of component: $d\mathbf{r}=[dx,dy,dz]$

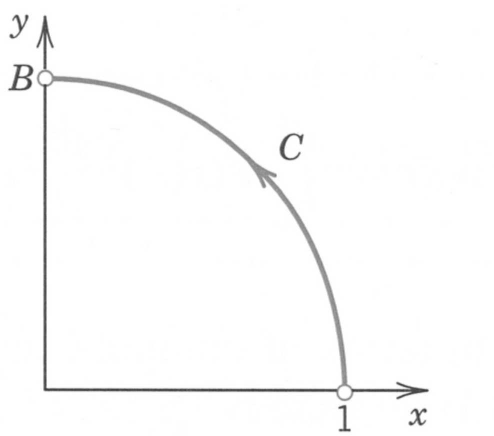
$\Rightarrow \int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}=\int_C (F_1dx+F_2dy+F_3dz)=\int_a^b (F_1x'+F_2y'+F_3z')\,dt$

For **closed curves**: $\oint_C$

IMPORTANT: the integral is **a scalar** since $\mathbf{F}\cdot\dfrac{d\mathbf{r}}{dt}$ with $t=s$ the arc length of C is the tangential component of $\mathbf{F}$

USAGE in mechanics: Work done by force **F** along path C

Ex. 1 Line integral in a plane



$$\mathbf{F}(\mathbf{r})=[-y,-xy]=-y\mathbf{i}-xy\mathbf{j}$$

Integrating on **circular arc**: $\mathbf{r}(t)=\cos t\ \mathbf{i}+\sin t\ \mathbf{j}$, where $0\le t\le \dfrac{p}{2}$

$$\Rightarrow \mathbf{r}'(t)=-\sin t\ \mathbf{i}+\cos t\ \mathbf{j}$$

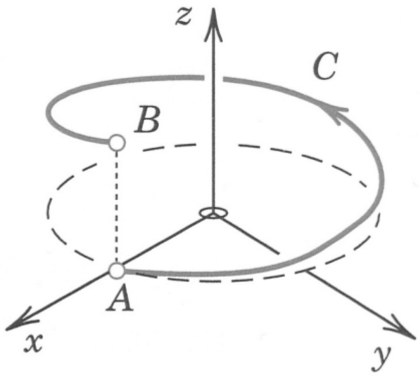
Since on $C \Rightarrow x(t)=\cos t \qquad y(t)=\sin t \ \Rightarrow \mathbf{F}(\mathbf{r})=-\sin t\ \mathbf{i}-\cos t\sin t\ \mathbf{j}$

$$\Rightarrow \int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}=\int_0^{p/2}(-\sin t\ \mathbf{i}-\cos t\sin t\ \mathbf{j})\cdot(-\sin t\ \mathbf{i}+\cos t\ \mathbf{j})\,dt=$$

Using $\int \sin^2 u\,du=\dfrac{u}{2}-\dfrac{\sin 2u}{4}$ and putting $u=\cos t\Rightarrow d\left(u^3\right)=3u^2du=3\cos^2 t(-\sin t)\,dt$

$$=\int_0^{p/2}\left(\sin^2 t-\cos^2 t\sin t\right)dt=\left.\frac{t}{2}-\frac{\sin 2t}{4}\right|_0^{\frac{p}{2}}-\left.\left(-\frac{1}{3}\cos^3\right)\right|_0^{\frac{p}{2}}=\frac{p}{4}-\frac{1}{3}\approx 0.4521$$

Ex. 2 Line integral in space



Integrate $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ on path = **helix**: $\vec{r} = \cos t\ \vec{i} + \sin t\ \vec{j} + 3t\ \vec{k}$, where $0 \le t \le 2p$

$$\vec{F}\cdot\vec{r}' = \left(3t\ \hat{i} + \cos t\ \hat{j} + \sin t\ \hat{k}\right)\cdot\left(-\sin t\ \hat{i} + \cot t\ \hat{j} + 3\ \hat{k}\right) = -3t\sin t + \cos^2 t + 3\sin t$$

The first integral using: $\int x\sin ax = \dfrac{\sin ax}{a^2} - \dfrac{x\cos ax}{a}$ (or integrating by part)

Using $\int \cos^2 u du = \dfrac{u}{2} + \dfrac{\sin 2u}{4}$ for the second integral

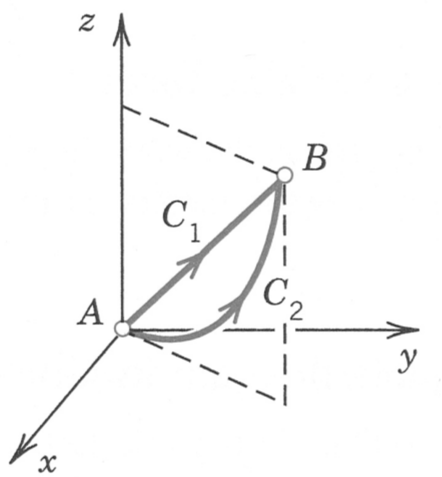
$$\int_0^{2p}\left(-3t\sin t + \cos^2 t + 3\sin t\right)dt = -3\left(\sin t - t\cos t\right)\Big|_0^{2p} + \left(\frac{t}{2} + \frac{\sin 2t}{4}\right)\Bigg|_0^{2p} + 3\cos t\Big|_0^{2p} =$$

$$= 6p + p + 0 = 7p \approx 21.99$$

Important points to consider:

1. Choice of representation of C does not matter. Value of integral is the same ➔ choose representation that will simplify integral

2. Choice of path is important. In general, value of the integral depends on path along which we integrate from a to b



Ex. 3 Dependence of line integral on path

$\mathbf{F}(\mathbf{r}) = [5z, xy, x^2 z] = 5z\mathbf{i} + xy\mathbf{j} + x^2 z\mathbf{k}$

a) straight line: $r_1(t) = [t,t,t] = t\,\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$

b) parabolic arc: $r_2(t) = [t,t,t^2] = t\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$, $0 \le t \le 1$

The first integral gives: $\int_0^1 (5t + t^2 + t^3)dt = \frac{37}{12}$

The second: $\int_0^1 (5t^2 + t^2 + 2t^5)dt = \frac{28}{12}$

Work done by a force

If $\mathbf{F}$ is a force and t is the time $\Rightarrow \dfrac{d\mathbf{r}}{dt} = \mathbf{v}$ is the velocity

The **work** done y the force: $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int \mathbf{F}\left(\mathbf{r}(t)\right)\cdot \mathbf{v}(t)\,dt$

Newton's 2nd law: $\mathbf{F} = m\mathbf{r}'' = m\mathbf{v}' \Rightarrow W = \int_a^b m\mathbf{v}'\cdot\mathbf{v}dt =$

Since $(v\cdot v)' = v'\cdot v + v\cdot v' = 2v'\cdot v \Rightarrow W = \int_a^b m\left(\dfrac{\mathbf{v}\cdot\mathbf{v}}{2}\right)' dt = \dfrac{1}{2}m|\mathbf{v}|^2\Big|_{t=a}^{t=b}$

$$\Rightarrow W = \int_a^b m\mathbf{v}'\cdot\mathbf{v}dt = \int_a^b m\left(\frac{\mathbf{v}\cdot\mathbf{v}}{2}\right)' dt = \frac{1}{2}m|\mathbf{v}|^2\Big|_{t=a}^{t=b}$$

Since **Kinetic energy** $K = \dfrac{1}{2}m|\mathbf{v}|^2$ then $W = \Delta K$

Other forms of line integral

Special cases when $\vec{F} = F_1\hat{i}$ or $\vec{F} = F_2\hat{j}$ or $\vec{F} = F_3\hat{i}$

$\int_C F_1 dx$ and $\int_C F_2 dy$ and $\int_C F_3 dz$ respectively

Special case when $\vec{F} = F_1\hat{i}$ and $F_1 = \dfrac{f}{(dx/dt)} \Rightarrow f = F_1 x'$

$$\int_C f(\vec{r})\,dt = \int_a^b f\left(\vec{r}(t)\right)dt$$

Ex. 4 $f = \left(x^2 + y^2 + z^2\right)^2$ integrated on the helix: $r = [\cos t, \sin t, 3t]$

This is a a special case $\int_C f(\vec{r})\,dt = \int_a^b f(\vec{r}(t))\,dt$ ➔ need to evaluate f on the helix

Now on the helix the function $f = \left(x^2 + y^2 + z^2\right)^2 = \left(\cos^2 t + \sin^2 t + 9t^2\right)^2 = \left(1 + 9t^2\right)^2$

$$\int_C f(\vec{r})\,dt = \int_0^{2p} \left(1 + 9t^2\right)^2 dt = \int_0^{2p} \left(1 + 18t^2 + 81t^4\right) dt =$$

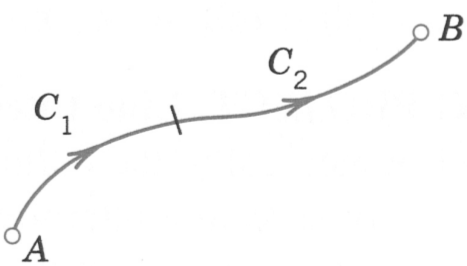
$$= t + 6t^3 + \frac{81}{5}t^5 \Big|_0^{2p} = 2p + 6(2p)^3 + \frac{81}{5}(2p)^5 \approx 160135$$

General properties

1. $\int_C k\vec{F} \cdot d\vec{r} = k \int_C \vec{F} \cdot d\vec{r}$

2. $\int_C \left(\vec{F} + \vec{G}\right) \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_C \vec{G} \cdot d\vec{r}$

3. $\int_C \vec{F} \cdot d\vec{r} = \int_{C1} \vec{F} \cdot d\vec{r} + \int_{C2} \vec{F} \cdot d\vec{r}$



4. $\int_a^b \vec{F} \cdot d\vec{r} = -\int_b^a \vec{F} \cdot d\vec{r}$

Th. 1 Direction preserving transformations of parameters

Any representation of C that give the same positive direction on C also yield the same value of the line integral

Line integral and independence of path

Th. 1 Independence of path

A line integral with continuous functions F_1, F_2, F_3 in domain D in space is independent of the path in D if and only if $\vec{F} = [F_1, F_2, F_3]$ is the gradient of some function f in D

$$F = \vec{\nabla} f \Rightarrow F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y} \text{ and } F_3 = \frac{\partial f}{\partial z}$$

In this case $\int_A^B F_1 dx + F_2 dy + F_3 dz = f(B) - f(A)$

Since $\vec{\nabla} f = \vec{F}$ ➔ f is a potential of $\vec{F}$ ➔ line integral is independent of the path only if $\vec{F}$ is the gradient of a potential

Ex. 1 Independence of path

$$I = \int_C 3x^2 dx + 2yzdy + y^2 dz \text{ from } A:(0,1,2) \text{ to } B:(1,-1,7)$$

Look for a potential:

$$f_x = 3x^2 \Rightarrow f = x^3 + g(y,z)$$
$$f_y = g_y = 2yz \Rightarrow g = y^2 z + h(z)$$
$$f_z = g_z = y^2 + h' = y^2 \Rightarrow h(z) = 0$$
$$\Rightarrow f = x^3 + y^2 z$$

Line integral $I = \int_C 3x^2 dx + 2yzdy + y^2 dz = x^3 + y^2 z\Big|_{0,1,2}^{1-1,7} = 1 + 7 - (0 + 2) = 6$

Integration around close path and independence of path

Th. 2 Independence of path

An integral is independent of the path in a domain D if and only if its value around every close path in D is zero

Mechanics

In the case of a force field derived from a potential, the force is said to be **conservative** because the work done by this force during a displacement from A and back to A is zero ➔ **energy is conserved**

Kinetic energy is the ability to do work by virtue of motion

Potential energy is ability to do work by virtue of position

If a body moves in a conservative field then after completion of round trip the kinematical + potential energy are restored

Friction + air resistance + water resistance = **dissipative forces** ➔ conversion of mechanical energy (vibration + noise) into heat

Exactness and independence of path

Differential form = basis for independence of path

Integrand: $F_1 dx + F_2 dy + F_3 dz$

This is exact if $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = F_1 dx + F_2 dy + F_3 dz$

Equivalent to: $\vec{F} = \vec{\nabla} f$

Topology: a domain D is called **simply connected** if every closed curve in D can be continuously shrank to any point in D without leaving D

Th. 3 Criterion for exactness and independence of path

$\vec{F}$ in line integral is continuous + continuous partial derivative in D

If line integral is independent of path ➔ integrand = exact differential

a) $\vec{\nabla} \times \vec{F} = 0$

$$\Rightarrow \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} \quad \text{and} \quad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \quad \text{and} \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$

b) If D simply connected and $\vec{\nabla} \times \vec{F} = 0$ then the integral is independent of the path

Ex. 2

$$I = \int_C 2xyz^2 dx + \left(x^2 z^2 + z\cos yz\right)dy + \left(2x^2 yz + y\cos yz\right)dz$$

To verify that the integral is independent of the path ➔ verify that integrand = exact differential

$$\Rightarrow \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} \Rightarrow \begin{matrix} (F_3)_y = 2x^2 z + \cos yz - yz\sin yz \\ (F_2)_z = 2x^2 z + \cos yz - yz\sin yz \end{matrix} \text{ OK!}$$

$$\Rightarrow \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \Rightarrow \begin{matrix} (F_1)_z = 4xyz \\ (F_3)_x = 4xyz \end{matrix} \text{ OK!}$$

$$\Rightarrow \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \Rightarrow \begin{matrix} (F_1)_y = 2xz^2 \\ (F_2)_x = 2xz^2 \end{matrix} \text{ OK!}$$

To find f :

$$\int F_2 dy = x^2 z^2 y + \sin yz + g(x,z) = f$$

$$f_x = 2xyz^2 + g_x = 2xyz^2 \Rightarrow g = h(z)$$

$$f_z = 2x^2 zy + y\cos yz + h' = 2x^2 zy + y\cos yz \Rightarrow h' = 0$$

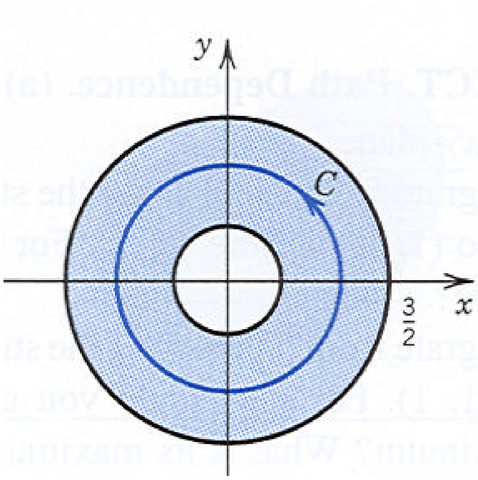
$$f = 2x^2 zy + \sin yz$$

Line integral from $(0,0,1)$ to $\left(1, \frac{p}{4}, 2\right)$: $f\left(1, \frac{p}{4}, 2\right) - f(0,01) = p + \sin\frac{p}{2} - 0 = p + 1$

Ex. 3 Connectedness

$$F_1 = -\frac{y}{x^2+y^2},\ F_2 = \frac{x}{x^2+y^2},\ F_3 = 0$$

Defined everywhere on plane except at the origin



Domain $D: \frac{1}{2} < \sqrt{x^2+y^2} < \frac{3}{2}$ ➔ not simply connected

Exact differential:

$$(F_2)_x = \frac{1}{x^2+y^2} - \frac{x(2x)}{\left(x^2+y^2\right)^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

$$(F_1)_y = -\frac{1}{x^2+y^2} + \frac{y(2y)}{\left(x^2+y^2\right)^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}$$

If integral $I = \int \frac{-ydx + xdy}{x^2+y^2}$ were independent of path the answer would have been zero

Integrating over circle $x^2+y^2=1$

Transformation: $x = r\cos\theta \Rightarrow dx = -r\sin\theta\, d\theta$ and $y = r\sin\theta \Rightarrow dy = r\cos\theta\, d\theta$

$$\Rightarrow I = \frac{1}{r^2}\int_0^{2\pi} r^2 d\theta = \int_0^{2\pi} d\theta = 2\pi \neq 0$$ not independent of the path