Linear algebra

Basic Concepts

Development of engineering ➔ use of computer + linear algebra

Matrix: rectangular array of numbers (**entries**) or functions (**elements**) enclosed in brackets

Ex. 1 Coefficient Matrix

$$\begin{aligned} 5x - 2y + z &= 0 \\ 3x + 47z &= 0 \end{aligned} \qquad \Rightarrow \mathbf{A} = \begin{pmatrix} 5 & -2 & 1 \\ 3 & 0 & 47 \end{pmatrix}$$

General notation: $\mathbf{A} = \left[a_{jk} \right] = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{j1} & \cdots & a_{jk} \end{pmatrix}$

Equality of matrices: $\mathbf{A} = \mathbf{B} \Rightarrow a_{jk} = b_{jk}$

Main diagonal: $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$

Vectors: **row vector** $\mathbf{a} = [a_1 \quad a_2 \quad \cdots \quad a_n]$ **column vector** $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Transposition: $\mathbf{a}^T \rightarrow$ column vector **,** $\mathbf{b}^T \rightarrow$ row vector

If $\mathbf{A} = \left[a_{jk} \right]$ then $\mathbf{A}^T = \left[a_{kj} \right]$ and a $m \times n$ matrix transposes into a $n \times m$ one

Different matrices

Rectangular matrices: $m \times n$

Square matrices: $n \times n$

Symmetric matrix: $\mathbf{A}^T = \mathbf{A}$ (obviously $n \times n$ matrix)

Skew-symmetric matrix: $\mathbf{A}^T = -\mathbf{A} = \left[-a_{jk}\right]$

Null matrix: $\mathbf{0} = \left[0_{jk}\right]$

Identity matrix: $\mathbf{1} = \left[d_{jk}\right]$ where $\delta_{jk} = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ if } j \neq k \end{cases}$ is **Knonecker delta**

Operations

Addition: $\mathbf{A} + \mathbf{B} = \mathbf{C} = \left[a_{jk} + b_{jk}\right]$ and $\left(\mathbf{A} + \mathbf{B}\right)^T = \mathbf{A}^T + \mathbf{B}^T$

Scalar multiplication: $c\mathbf{A} = \left[ca_{jk}\right]$ and $\left(c\mathbf{A}\right)^T = c\mathbf{A}^T$

General rules:

Addition of matrices	Scalar multiplication
a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	e) $c\left(\mathbf{A} + \mathbf{B}\right) = c\mathbf{A} + c\mathbf{B}$
b) $\left(\mathbf{A} + \mathbf{B}\right) + \mathbf{C} = \mathbf{A} + \left(\mathbf{B} + \mathbf{C}\right)$	f) $\left(c + k\right)\mathbf{A} = c\mathbf{A} + k\mathbf{A}$
c) $\mathbf{A} + \mathbf{0} = \mathbf{A}$	g) $c\left(k\mathbf{A}\right) = \left(ck\right)\mathbf{A}$
d) $\mathbf{A} - \mathbf{A} = \mathbf{0}$	h) $1\mathbf{A} = \mathbf{A}1 = \mathbf{A}$

Matrix multiplication

Matrix multiplication $\mathbf{C} = \mathbf{A}\mathbf{B}$, where $\mathbf{A} = \left[a_{jk}\right]$ is $m \times n$ matrix and $\mathbf{B} = \left[b_{jk}\right]$ is a $r \times p$ matrix, is defined if and only if $n = r \Rightarrow C$ is a $m \times p$ matrix where:

$$c_{jk} = \sum_{l=1}^{n} a_{jl} b_{lk} \qquad \begin{Bmatrix} j = 1, \ldots, m \\ k = 1, \ldots, p \end{Bmatrix}$$

Multiplication of rows of A by columns of B

$$\mathbf{AB} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \boxed{a_{j1} \quad a_{j2} \quad \cdots \quad a_{jn}} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & \left[b_{1k}\right. & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2k} & \cdots & b_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & \left.b_{nk}\right] & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1k} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2k} & \cdots & c_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{j1} & c_{j2} & \cdots & \boxed{c_{jk}} & \cdots & c_{jp} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mk} & \cdots & c_{mp} \end{pmatrix} = \mathbf{C}$$

Ex. $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow \mathbf{AB} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 1\cdot 3 & 1\cdot 2 + 1\cdot 4 \\ 1\cdot 1 + 1\cdot 3 & 1\cdot 2 + 1\cdot 4 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 4 & 6 \end{pmatrix}$

Transpose of product: $\left(\mathbf{AB}\right)^T = \mathbf{B}^T \mathbf{A}^T$

Consequences of the definition

a) $\mathbf{AB} \neq \mathbf{BA}$

Order of multiplication is important:

B Premultiplied (multiplied from left) by **A**

A Postmultiplied (multiplied from right) by **B**

In the example above: $\mathbf{BA} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1\cdot1+2\cdot1 & 1\cdot1+2\cdot1 \\ 3\cdot1+4\cdot1 & 3\cdot1+4\cdot1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 7 & 7 \end{pmatrix} \Rightarrow \mathbf{AB} \neq \mathbf{BA}$

b) $\mathbf{AB} = \mathbf{0} \not\Rightarrow \mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$ or $\mathbf{BA} = \mathbf{0}$

c) $\mathbf{AB} = \mathbf{AD} \not\Rightarrow \mathbf{B} = \mathbf{D}$

d) $(k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B})$

e) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

f) $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$

g) $\mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA}+\mathbf{CB} \neq (\mathbf{A}+\mathbf{B})\mathbf{C}$

Special matrices

Upper triangular matrices: Ex. $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ or $\begin{pmatrix} 4 & 2 & 2 & 0 \\ 0 & -3 & 5 & 1 \\ 0 & 0 & -1 & -6 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

Lower triangular matrices: Ex. $\mathbf{B} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ or $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 6 \end{pmatrix}$

Transpose of Upper triangular = Lower triangular matrix: $\mathbf{A}^T = \mathbf{B}$ and $\mathbf{B}^T = \mathbf{A}$

Diagonal matrices: Ex. $\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$

Scalar matrix: Ex. $\mathbf{S} = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \Rightarrow \mathbf{S\,A} = \mathbf{A\,S} = c\mathbf{A}$

Unit matrix: Ex. $\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \mathbf{1\,A} = \mathbf{A\,1} = \mathbf{A}$

Inner product of vectors = dot product

If $\mathbf{a}$ and $\mathbf{b}$ are a row and column matrices then $\mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum_{l=1}^{n} a_l b_l$

Multiplication of matrices: every entry of $\mathbf{C} = \mathbf{A\,B}$ is an inner product $c_{jk} = a_j \cdot b_k$

Gauss elimination

Most important uses of matrices = solving linear systems of equations

Consider a system of m equations with n unknowns $x_1, x_2, \ldots, x_n$

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

Coefficients: a_{jk} this is a matrix

Solution is a vector: $\mathbf{x}$ components = solutions

Trivial solution: $\mathbf{x} = \mathbf{0}$

Matrix form: $\mathbf{Ax} = \mathbf{b}$

$$
\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}
$$

Homogeneous: $b_k = 0$

Augmented matrix: $\tilde{\mathbf{A}} = \begin{bmatrix} a_{jk} & b_k \end{bmatrix}$

$$
\begin{pmatrix} a_{11} & \ldots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \ldots & a_{mn} & b_n \end{pmatrix}
$$

Th. 1 Row-equivalent systems

Row-equivalent linear systems have same sets of solutions

Elementary row operations ➔ produce **row-equivalent systems**

a) Interchanges of 2 rows

b) Addition of constant multiple of one row with another

c) Multiplication of row by non zero constant

3 possible linear systems: $\begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_n \end{pmatrix}$

1. **Overdetermined**: more equations than unknowns $m > n$
2. **Determined**: $m = n$
3. **Underdetermined**: less equations than unknowns $m < n$

Consistent system ➔ exist at least one solution

Inconsistent system ➔ no solution

Ex. 1 - overdetermined system

$$
\begin{array}{llll}
x_1 & -x_2 & +x_3 & =0 \\
-x_1 & +x_2 & -x_3 & =0 \\
 & 10x_2 & +25x_3 & =90 \\
20x_1 & +10x_2 & & =80
\end{array}
$$

Matrix form:

$$
\begin{array}{r}
pivot\ 1=1 \\ \\ \\ \\
\end{array}
\begin{pmatrix}
1 & -1 & 1 & 0 \\
-1 & 1 & -1 & 0 \\
0 & 10 & 25 & 90 \\
20 & 10 & 0 & 80
\end{pmatrix}
\Rightarrow
\begin{array}{r}
\\ row2+row1 \\ \\ row4-20row1
\end{array}
\begin{pmatrix}
1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 10 & 25 & 90 \\
0 & 30 & -20 & 80
\end{pmatrix}
$$

$$
\begin{array}{r}
pivot\ 2=10 \\
\Rightarrow row2 \rightleftarrows row3 \\
row3 \rightleftarrows row4 \\
-3row2+row3
\end{array}
\begin{pmatrix}
1 & -1 & 1 & 0 \\
0 & 10 & 25 & 90 \\
0 & 30 & -20 & 80 \\
0 & 0 & 0 & 0
\end{pmatrix}
\Rightarrow
\begin{array}{r}
\\ pivot\ 2=10 \\ row3-3row2 \\ pivot\ 3=-95
\end{array}
\begin{pmatrix}
1 & -1 & 1 & 0 \\
0 & 10 & 25 & 90 \\
0 & 0 & -95 & -190 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

Back substitutions yield:

$$x_3=\frac{-190}{-95}=2$$

$$\Rightarrow 10x_2+25x_3=90 \Rightarrow x_2=4$$

$$\Rightarrow x_1-x_2+x_3=0 \Rightarrow x_1=2$$

Ex. 2 underdetermined system

$$\begin{pmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{pmatrix}$$

$$\begin{pmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{pmatrix} \begin{matrix} pivot1 = 3.0 \\ -0.2row1 + row2 \\ -0.4row1 + row3 \end{matrix}$$

$$\begin{pmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ pivot2 = 1.1 \\ row2 + row3 \end{matrix}$$

Undetermined ➔ infinity of solutions

Echelon form

$$\begin{array}{lllll} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & = b_1 \\ & c_{22}x_2 & +\cdots & +c_{2n}x_n & = b_2 \\ & & & & \vdots \\ & +k_{rr}x_r & +\cdots & +k_{rn}x_n & = \tilde{b}_r \\ & & & 0 & = \tilde{b}_{r+1} \\ & & & \vdots & \vdots \\ & & & 0 & = \tilde{b}_m \end{array}$$

No solution $\Rightarrow r < m$ and one of number $\tilde{b}_{r+1}, \ldots, \tilde{b}_m \neq 0$

Precisely one solution $\Rightarrow r = n$ and $\tilde{b}_{r+1}, \ldots, \tilde{b}_m = 0$ if present

Infinitely many solutions $\Rightarrow r < n$ and $\tilde{b}_{r+1}, \ldots, \tilde{b}_m = 0$ if present

Rank of Matrix, linear independence and vector space

Linear combination: given any set m of vector $\mathbf{a}_{(1)},\ldots,\mathbf{a}_{(m)}$ with the same number of components a linear combination is of the form $c_1\mathbf{a}_{(1)}+\cdots+c_m\mathbf{a}_{(m)}$

Considering equation: $c_1\mathbf{a}_{(1)}+\cdots+c_m\mathbf{a}_{(m)}=\mathbf{0}$

The vectors are linearly independent if the coefficients $c_1=c_2=\cdots=c_m=0$

Ex. 1
$$\begin{aligned}\mathbf{a}_{(1)}&=\begin{bmatrix}3 & 0 & 2 & 2\end{bmatrix}\\ \mathbf{a}_{(2)}&=\begin{bmatrix}-6 & 42 & 24 & 54\end{bmatrix}\\ \mathbf{a}_{(3)}&=\begin{bmatrix}21 & -21 & 0 & -15\end{bmatrix}\end{aligned}$$

Linearly dependent because: $6\mathbf{a}_{(1)}-\frac{1}{2}\mathbf{a}_{(2)}-\mathbf{a}_{(3)}=0$

However, $c_1\mathbf{a}_{(1)}+c_2\mathbf{a}_{(2)}=0\Rightarrow c_1=c_2=0$

Rank of matrix $\mathbf{A}=\left[a_{jk}\right]$ = maximum number of linearly independent row vectors forming the matrix ➔ Rank $\mathbf{A}=0$ if and only if $\mathbf{A}=\mathbf{0}$

Th. 1 Rank in terms of column vectors

Rank of $\mathbf{A}$ equals the maximum number of linearly independent column vectors forming $\mathbf{A}$

$\Rightarrow \mathbf{A}$ and $\mathbf{A}^T$ have the same rank

Vector space: non empty set $\mathbf{V}$ of vectors such that with any vectors $\mathbf{a}_{(1)},\dots,\mathbf{a}_{(m)}$ in $\mathbf{V}$ all their linear combinations $c_1\mathbf{a}_{(1)}+\cdots+c_m\mathbf{a}_{(m)}$ are elements of $\mathbf{V}$ and satisfy laws of Matrix

Dimension of $\mathbf{V}$: number of linearly independent vectors in $\mathbf{V}$

Basis: linearly independent set of vectors in $\mathbf{V}$ consisting of maximum possible of number of vectors $\Rightarrow$ number of vectors of a basis for $\mathbf{V}$ equals dimension of $\mathbf{V}$

Span of $\mathbf{V}$: set of all linear combinations of given vectors $\mathbf{a}_{(1)},\dots,\mathbf{a}_{(p)}$ with same number of components $\Rightarrow$ span = vector space

Subspace: non empty subset of $\mathbf{V}$ that itself form a vector space

Ex. 2 The span of the 3 vectors in example 1 is a vector space of dimension 2, and a basis is $\mathbf{a}_{(1)},\mathbf{a}_{(2)}$, for instance, or $\mathbf{a}_{(1)},\mathbf{a}_{(3)}$ etc.

Row space of $\mathbf{A}$: span of row vectors

Column space of $\mathbf{A}$: span of column vectors

Th. 2 row space and column space

Row space and column space of $\mathbf{A}$ have same dimension = rank of $\mathbf{A}$

Th. 3 row-equivalent matrices

Row-equivalent matrices have the same rank

To determine rank of $\mathbf{A}$ reduce $\mathbf{A}$ to echelon form

Ex. 3

$$\mathbf{A}=\begin{bmatrix}3&0&2&2\\-6&42&24&54\\21&-21&0&-15\end{bmatrix}\Rightarrow\begin{array}{r}\\ row2+2row1\\ row3-7row1\end{array}\begin{bmatrix}3&0&2&2\\0&42&28&58\\0&-21&-14&-29\end{bmatrix}\Rightarrow\begin{bmatrix}3&0&2&2\\0&42&28&58\\0&0&0&0\end{bmatrix}_{row3+\frac{1}{2}row2}$$

Rank of $\mathbf{A}$ is 2

Th. 4 linear dependence and independence

p vectors $\mathbf{x}_{(1)},\ldots,\mathbf{x}_{(p)}$ (with *n* components each) are linearly independent if the matrix with row vectors $\mathbf{x}_{(1)},\ldots,\mathbf{x}_{(p)}$ has rank *p*; they are linearly dependent if that rank is less than *p*

Th. 5

p vectors with $n < p$ components are always linearly dependent

For instance, 3 or more vectors in a plane are always linearly dependent. Similarly, four or more vectors in space are linearly dependent

Th. 6

The vector space R^n consisting of all vectors with *n* components has dimension *n*

Solutions of linear systems: existence, uniqueness, general form

Submatrix of $\mathbf{A}$: matrix obtained from $\mathbf{A}$ by omitting some rows or columns (or both) – this includes $\mathbf{A}$ itself

Th. 1 Fundamental theorem for linear systems

a) **Existence**: a linear system of m equations with n unknowns $x_1, x_2, \ldots, x_n$ has a solution if and only if the coefficient matrix $\mathbf{A}$ and augmented matrix $\tilde{\mathbf{A}}$ have the same rank

b) **Uniqueness**: the system has precisely one solution if and only if this common rank $k = n$

c) **Infinitely many solutions**: if rank $r < n \Rightarrow$ infinitely many solutions

d) **Gauss elimination**: if solution exists it can be obtained by Gauss elimination$\Rightarrow$such elimination can reveal the rank

Th. 2 homogenous linear systems

- An homogenous linear system has one trivial solution $x_1 = 0, \ldots, x_n = 0$
- Non trivial solutions exist if and only if rank $\mathbf{A} < n$
- If rank $\mathbf{A} = r < n$ these solutions plus the trivial one form a vector space of dimension $n - r \Rightarrow$ **solution space**
- In particular any linear combination of solution vectors is also a solution

Solution space of $\mathbf{A}$ also called the **null space**, because $\mathbf{Ax} = \mathbf{0}$

$\Rightarrow$ rank $\mathbf{A}$ = nullity $\mathbf{A} = n$

Th. 3 homogenous linear systems with fewer equations than unknowns

This system always has nontrivial solutions

Th. 4 nonhomogenous linear systems

If such a system has solutions, then all these solutions are of the form $x = x_0 + x_h$ where x_0 is any fixed solution and x_h runs through all solution of corresponding homogeneous system

Determinants and Cramer's rule

Originally introduced to solve linear system

Turned out to be impractical in computation (Gauss elimination method used instead)

Application fundamental for eigenvalue problems

Nth-order determinant: expression associated with $n\times n$ matrix $\mathbf{A}=\left[a_{jk}\right]$

2nd order determinant: $D=\det\mathbf{A}=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$

Ex. 1 Cramer's rule for linear systems of 2 equations

For a linear system of 2 equations with 2 unknowns:

$$a_{11}x_1+a_{12}x_2=b_1$$
$$a_{21}x_1+a_{22}x_2=b_2$$

Eliminating $x_2\Rightarrow\left(a_{11}a_{22}-a_{12}a_{21}\right)x_1=b_1a_{22}-a_{12}b_2$

Eliminating $x_1\Rightarrow\left(a_{11}a_{22}-a_{12}a_{21}\right)x_2=a_{11}b_2-b_1a_{21}$

This is equivalent to:

$$x_1=\frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} \quad \text{and} \quad x_2=\frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D}$$

Ex. 1

$$\begin{aligned} 4x_1+3x_2&=12 \\ 2x_1+5x_2&=-8 \end{aligned} \Rightarrow x_1=\frac{\begin{vmatrix} 12 & 3 \\ -8 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}}=\frac{84}{14}=6 \text{ and } \Rightarrow x_2=\frac{\begin{vmatrix} 4 & 12 \\ 2 & -8 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}}=\frac{-56}{14}=-4$$

If the system is homogeneous and $D\neq 0$ it *has only the trivial solution* $x_1=x_2=0$, and if $D=0$, it also has non trivial solutions (eigenvalue problem)

3$^{\text{rd}}$ **order determinant**: $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Ex. 2 Cramer's rule for linear systems of 3 equations

For a linear system of 3 equations with 3 unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13} = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23} = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33} = b_3 \end{aligned}$$

$$x_1 = \frac{D_1}{D} \qquad x_2 = \frac{D_2}{D} \qquad x_3 = \frac{D_3}{D}$$

Where:

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \qquad D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \qquad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{23} & a_{32} & b_3 \end{vmatrix}$$

nth order determinant

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

For $n = 1$, $D = a_{11}$

For $n \geq 2$

$$D = a_{j1}C_{j1} + a_{j2}C_{j2} + \ldots + a_{jn}C_{jn} \quad j = 1,2,\ldots,n$$

or using the columns:

$$D = a_{1k}C_{1k} + a_{2k}C_{2k} + \ldots + a_{nk}C_{nk} \quad k = 1,2,\ldots,n$$

Where $C_{jk} = (-1)^{j+k} M_{jk}$ = **cofactor** of a_{jk} in D

Where M_{jk} is the determinant of $n-1$ order = the **minor** of a_{jk} in D

IMPORTANT: we may expand D using any row or column

Alternative notation:

$$D = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} M_{jk}$$

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk}$$

Ex. 3 for $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \Rightarrow M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix},\ M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix},\ M_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

Ex. 4 for $D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} = -12$

General properties of determinants

Th. 1 Behavior of determinant under elementary row operations

a) Interchange of 2 rows $\Rightarrow -1D$

b) Add multiple of one row to another $\Rightarrow D$

c) Multiplication of one row by $c \Rightarrow cD$

Simplification: since determinant of triangular matrix = multiplication of diagonal terms
$\Rightarrow$ reducing matrix to triangular form simplify calculation

Th. 2 Further properties

d) $D(\mathbf{A}^T) = D$

e) Matrix with a zero row or column $\Rightarrow D = 0$

f) Proportional (or equal) rows or columns $\Rightarrow D = 0$

Th. 3 Rank in terms of determinants

An $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ has rank $r \geq 1$ if and only if $\mathbf{A}$ has an $r \times r$ submatrix with nonzero determinant

In particular, if $\mathbf{A}$ is a $n \times n$ matrix, rank $\mathbf{A} = n$ if and only if det $\mathbf{A} \neq 0$

Cramer's rule

Th. 4 Cramer's theorem (solution of linear systems by determinants)

a) If a linear system of n equations of n unknowns has a non zero coefficient determinant, the system has precisely one solution:

$$x_1 = \frac{D_1}{D},\ x_2 = \frac{D_2}{D}, \dots,\ x_n = \frac{D_n}{D}$$

Where D_k = determinants obtained by replacing the kth column by the entries:

$b_1, b_2, \dots b_n$

b) If the system is homogenous and $D \neq 0$ then the only solution is the trivial one

c) If $D = 0$ there also exist a non trivial solution (eigenvalue problem)

Inverse of Matrix and Gauss-Jordan elimination

Works only for $n \times n$ matrices

Inverse of $\mathbf{A} = \mathbf{A}^{-1} \Rightarrow \mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{1}$

Nonsingular matrix $\Rightarrow \mathbf{A}^{-1}$ exists and is unique

Th.1 existence of inverse

The inverse exists if and only if rank of $\mathbf{A} = n \Rightarrow D \neq 0$

Gauss-Jordan method

We start with $\mathbf{AX} = \mathbf{I} \Rightarrow \mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{I} = \mathbf{A}^{-1}$, hence to solve $\mathbf{AX} = \mathbf{I}$ we can apply Gauss elimination to $\tilde{\mathbf{A}} = [\mathbf{AI}] \Rightarrow [\mathbf{UH}]$ where $\mathbf{U}$ is upper triangular

Gauss-Jordan elimination reduces $\mathbf{U}$ to diagonal form $\mathbf{I}$ transforming $\mathbf{H}$ into $\mathbf{K}$ such that $\Rightarrow [\mathbf{I}\ \mathbf{K}]$ which is the augmented matrix $\mathbf{IX} = \mathbf{K} \Rightarrow \mathbf{K} = \mathbf{A}^{-1}$

Ex. 1

$$\mathbf{A}=\begin{pmatrix}-1 & 1 & 2\\ 3 & -1 & 1\\ -1 & 3 & 4\end{pmatrix}\Rightarrow \tilde{\mathbf{A}}=\begin{pmatrix}-1 & 1 & 2 & 1 & 0 & 0\\ 3 & -1 & 1 & 0 & 1 & 0\\ -1 & 3 & 4 & 0 & 0 & 1\end{pmatrix}$$

First transformation

$$\begin{array}{r} \\ r2+3r1 \\ r3-r1 \end{array}\begin{pmatrix}-1 & 1 & 2 & 1 & 0 & 0\\ 0 & 2 & 7 & 3 & 1 & 0\\ 0 & 2 & 2 & -1 & 0 & 1\end{pmatrix}$$

Second transformation

$$\begin{array}{r} \\ \\ r3-r2 \end{array}\begin{pmatrix}-1 & 1 & 2 & 1 & 0 & 0\\ 0 & 2 & 7 & 3 & 1 & 0\\ 0 & 0 & -5 & -4 & -1 & 1\end{pmatrix}$$

Reducing **U** to **I**

$$\begin{array}{r} -r1 \\ 0.5r2 \\ -0.2r3 \end{array}\begin{pmatrix}1 & -1 & -2 & -1 & 0 & 0\\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0\\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2\end{pmatrix}$$

$$\begin{array}{r} r1+2r3 \\ r2-3.5r3 \\ \\ \end{array}\begin{pmatrix}1 & -1 & 0 & 0.6 & 0.4 & -0.4\\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7\\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2\end{pmatrix}$$

$$\begin{array}{r} r1+r2 \\ \\ \\ \end{array}\begin{pmatrix}1 & 0 & 0 & -0.7 & 0.2 & 0.3\\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7\\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2\end{pmatrix}$$

$$\Rightarrow \mathbf{A}^{-1}=\begin{pmatrix}-0.7 & 0.2 & 0.3\\ -1.3 & -0.2 & 0.7\\ 0.8 & 0.2 & -0.2\end{pmatrix}$$

Useful formulas for inverses

Th. 2 Inverse of non singular $n \times n$ matrix

$\mathbf{A}^{-1} = \frac{1}{D}\left[A_{jk}\right]^T$ where A_{jk} is the cofactor of a_{jk} in D

2 x 2 matrices

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \text{ and } A_{jk} = \begin{vmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{vmatrix}$$

$$\Rightarrow \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\left(A_{jk}\right)^T = \frac{1}{D}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Diagonal matrices

$A = \left[a_{jk}\right]$ where $a_{jk} = 0$ for $j \neq k$

The inverse exists if and only if $a_{jk} \neq 0$ then $\mathbf{A}^{-1}$ diagonal with entries $\frac{1}{a_{11}}, \ldots, \frac{1}{a_{nn}}$

Inverse of products

$(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}$ or $(\mathbf{ABC}\ldots\mathbf{PQ})^{-1} = \mathbf{P}^{-1}\mathbf{Q}^{-1}\ldots\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$

Th. 3 Cancellation law

Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be $n \times n$ matrices

a) If rank $\mathbf{A} = n$ then $\mathbf{AB} = \mathbf{AC} \Rightarrow \mathbf{B} = \mathbf{C}$

b) If rank $\mathbf{A} = n$ then $\mathbf{AB} = \mathbf{0} \Rightarrow \mathbf{B} = \mathbf{0}$

c) If $\mathbf{A}$ is singular so are $\mathbf{BA}$ and $\mathbf{AB}$

Determinant of products

$\det(\mathbf{AB}) = \det \mathbf{A} \cdot \det \mathbf{B}$

Matrix Eigenvalue problems

Let $\mathbf{A} = \left[a_{jk} \right]$ a $n \times n$ matrix

Vector equation: $\mathbf{Ax} = l\,\mathbf{x}$

Eigenvalue (characteristic): l = value for which $\mathbf{Ax} = l\,\mathbf{x}$ and $\mathbf{x} \neq \mathbf{0}$

$\Rightarrow$ $\mathbf{x}$ is an **eigenvector**

Set of eigenvalues of $\mathbf{A}$ = **spectrum** of $\mathbf{A}$

Largest of absolute eigenvalues of $\mathbf{A}$ = **spectral radius**

The set of all eigenvalues of **A + 0 = vector space**

Ex. 1

$$\mathbf{A} = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix} \Rightarrow \mathbf{Ax} = l\,\mathbf{x} \Rightarrow \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = l \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} -5x_1 + 2x_2 = l\,x_1 \\ 2x_1 - 2x_2 = l\,x_2 \end{matrix} \Rightarrow \begin{matrix} (-5 - l)\,x_1 + 2x_2 = 0 \\ 2x_1 + (-2 - l)\,x_2 = 0 \end{matrix}$$

Equivalent to $\Rightarrow (\mathbf{A} - l\,\mathbf{I})\,\mathbf{x} = \mathbf{0}$

By Cramer's rule (page 18 - part b), if $D(\mathbf{A} - l\,\mathbf{I}) = 0$ the system has a non trivial solution

$$D(l) = D(\mathbf{A} - l\,\mathbf{I}) = \begin{vmatrix} -5 - l & 2 \\ 2 & -2 - l \end{vmatrix} = l^2 + 7l + 6 = 0$$

where $D(l)$ is the **characteristic determinant** and $D(l) = 0$ is the **characteristic equation**

Solving $D(l) = 0$ ➔ Eigenvalues**:** $l = -1$ and $l = -6$

To find eigenvectors: substitute eigenvalues in $(\mathbf{A} - l\,\mathbf{I})\,\mathbf{x} = \mathbf{0}$:

For $l_1 = -1 \Rightarrow \begin{matrix} -4x_1 + 2x_2 = 0 \\ 2x_1 - x_2 = 0 \end{matrix} \Rightarrow x_2 = 2x_1$, chosing $x_1 = 1 \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $l_1 = -6 \Rightarrow \begin{matrix} x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{matrix} \Rightarrow x_2 = -\frac{1}{2}x_1$, chosing $x_1 = 2 \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Th. 1 Eigenvalues

Eigenvalues of square matrix $\mathbf{A}$ are the roots of the characteristic equation $D(l) = 0$

Hence $n \times n$ matrix has at least one eigenvalue and at most n numerically different eigenvalues

Th. 2 Eigenvectors

If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ with eigenvalue l so is $k\mathbf{x}$ with $k \neq 0$

Ex. 2

$$A=\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} \Rightarrow A-l\,I=\begin{pmatrix} -2-l & 2 & -3 \\ 2 & 1-l & -6 \\ -1 & -2 & -l \end{pmatrix}$$

$$\Rightarrow D(A-l\,I)=(-2-l)\begin{vmatrix} 1-l & -6 \\ -2 & -l \end{vmatrix}-2\begin{vmatrix} 2 & -6 \\ -1 & -l \end{vmatrix}-3\begin{vmatrix} 2 & 1-l \\ -1 & -2 \end{vmatrix}$$

$$\Rightarrow D(l)=0=-l^3-l^2+21l+45=0 \Rightarrow l_1=5 \quad l_2=l_3=-3$$

Applying Gauss elimination to $(\mathbf{A}-l\,\mathbf{I})\,\mathbf{x}=\mathbf{0}$ with $l_1=5 \Rightarrow \mathbf{x}_1=\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

For $l_2=-3 \Rightarrow (\mathbf{A}-l\,\mathbf{I})=\mathbf{A}+3\mathbf{I}=\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix}$

Which reduces to $\begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ a rank 1 matrix

$$\Rightarrow x_1=-2x_2+3x_3$$

Choosing $x_2=1$ and $x_3=0 \Rightarrow \mathbf{x}_2=\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

Choosing $x_2=0$ and $x_3=1 \Rightarrow \mathbf{x}_3=\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

Algebraic multiplicity: order M_l of eigenvalue as a root of characteristic polynomial

Geometric multiplicity: number m_l of linear independent eigenvector $\Rightarrow m_l$ =dimension of eigenspace

Since the characteristic polynomial has degree *n*, **sum of all algebraic multiplicity** = *n* the degree of the polynomial

In Ex. 2, for $l = -3 \Rightarrow m_l = M_l = 2$, but in general $m_l \leq M_l$

Difference = **defect**: $\Delta l = M_l - m_l$

Ex. for $l = -3 \Rightarrow \Delta l = 0$

In general $\Delta l > 0$

Ex. 3

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \det(\mathbf{A} - l\,\mathbf{I}) = \begin{vmatrix} -l & 1 \\ 0 & -l \end{vmatrix} = l^2 = 0$$

Hence $l = 0$ is eigenvalue with $M_l = 2$ but $m_l = 1$, since eigenvectors result from $-0x_1 + x_2 = 0 \Rightarrow x_2 = 0 \Rightarrow \Delta_0 = 1$

Ex. 4

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \det(\mathbf{A} - l\,\mathbf{I}) = \begin{vmatrix} -l & 1 \\ -1 & -l \end{vmatrix} = l^2 + 1 = 0$$

$\Rightarrow l_1 = i$ and $l_2 = -i$

eigenvectors result from $-ix_1 + x_2 = 0$ and $ix_1 + x_2 = 0$, choosing $x_1 = 1$

$$\Rightarrow \mathbf{x}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ and } \Rightarrow \mathbf{x}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Applications

Ex. 1 stretching of elastic membrane

Elastic membrane in $x_1x_2-\text{plane}$ with boundary conditions $x_1^2+x_2^2=1$ is stretched so that a point $P:(x_1,x_2)$ goes over into the point $Q:(y_1,y_2)$ as given by:

$$\mathbf{Ax}=\mathbf{y}\Rightarrow\begin{pmatrix}5&3\\3&5\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

Stretching ➔ simple transformation of coordinates $P:(x_1,x_2)\to Q:(y_1,y_2)$

Principal direction = vector **x** of P ➔ same or exactly opposite direction as **y** in Q

➔ Need to find eigenvectors

➔ Characteristic equations: $(5-l)^2-9=0\Rightarrow l_1=8$ and $l_2=2$

For $l_1=8\Rightarrow-3x_1+3x_2=0\Rightarrow x_1=x_2$ and $3x_1-3x_2=0\Rightarrow x_1=x_2=1$

For $l_2=2\Rightarrow 3x_1+3x_2=0$ and $3x_1+3x_2=0\Rightarrow x_1=-x_2$ choosing $x_1=1\Rightarrow x_2=-1$

$$\Rightarrow\mathbf{x}_1=\begin{pmatrix}1\\1\end{pmatrix}\text{ and }\Rightarrow\mathbf{x}_2=\begin{pmatrix}1\\-1\end{pmatrix}$$

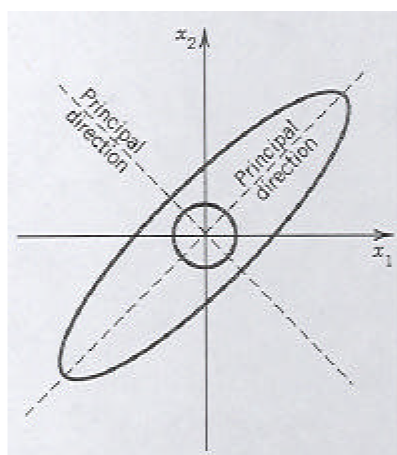
First = vector at 45° -- other = vector at 135°

Along principal directions ➔ membrane stretches by factor 8 and 2 respectively

Choosing principal directions as direction of new m_1m_2

Cartesian system of coordinates $m_1=r\cos j$ and $m_2=r\sin j$ $\Rightarrow$ Unstretched boundary $\cos j$ and $\sin j$

After stretching boundaries $z_1=8\cos j$ and $z_2=2\sin j$



Deformed boundary = ellipse: $\dfrac{z_1^2}{8^2}+\dfrac{z_2^2}{2^2}$ ➔ principal semi-axes 8 and 2 in principal direction

Ex. 2 vibrating system of 2 masses on two springs

$$y_1'' = -5y_1 + 2y_2$$
$$y_2'' = 2y_1 - 2y_2$$

Equivalent to $\mathbf{Y}'' = \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \mathbf{AY} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Trying $\mathbf{y} = \mathbf{x}e^{wt} \Rightarrow \mathbf{y}'' = w^2\mathbf{x}e^{wt} = \mathbf{A}\mathbf{x}e^{wt}$

Putting $w^2 = l \Rightarrow \mathbf{Ax} = l\,\mathbf{x}$

For $\mathbf{A} = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix} \Rightarrow (-5 - l)(-2 - l) - 4 = l^2 + 7l + 6 = 0$

$\Rightarrow l_1 = -1 \Rightarrow w = \sqrt{-1} = \pm i$ and $l_2 = -6 \Rightarrow w = \sqrt{-6} = \pm i\sqrt{6}$

Eigenvectors (section 6.1 Ex. 1): $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

4 complex solutions:

$$\mathbf{x}_1 e^{\pm it} = \mathbf{x}_1(\cos t \pm i \sin t)$$

$$\mathbf{x}_2 e^{\pm i\sqrt{6}t} = \mathbf{x}_2\left(\cos\sqrt{6}t \pm i\sin\sqrt{6}t\right)$$

Physically only real solutions are meaningful

General solution: $\mathbf{y} = \mathbf{x}_1(a_1 \cos t + b_1 \sin t) + \mathbf{x}_2\left(a_2 \cos\sqrt{6}t + b_2 \sin\sqrt{6}t\right)$

Describe harmonic oscillation of two masses

Symmetric, Skew-Symmetric and Orthogonal matrices

Symmetric matrix: $\mathbf{A}^T = \mathbf{A}$

Skew-Symmetric matrix: $\mathbf{A}^T = -\mathbf{A}$

Orthogonal matrix: $\mathbf{A}^T = \mathbf{A}^{-1}$

Th. 1 Eigenvalues of symmetric and skew-symmetric matrix

a) Eigenvalues of symmetric matrix are real

b) Eigenvalues of skew-symmetric matrix are pure imaginary or zero

Orthogonal transformation: $\mathbf{y} = \mathbf{A}\,\mathbf{x}$ for each $\mathbf{x}$ in $R^n \to \mathbf{y}$ in R^n

Ex. Matrix of rotation: $\mathbf{y} = \begin{pmatrix} \cos q & -\sin q \\ \sin q & \cos q \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Th. 2 Invariance of inner product

Orthogonal transformations preserve values of inner product

$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}$

➔ preserves **norm** of vector: $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{\mathbf{a}^T \mathbf{a}}$

Th. 3 Orthogonality of column and row vectors

A real $n \times n$ matrix is orthogonal if and only if the column vectors form an orthogonal system

$$a_j \cdot a_k = a_j^T a_k = \begin{cases} 0 \text{ for } j \neq k \\ 1 \text{ for } j = k \end{cases}$$

Th. 4 Determinant of orthogonal matrix

$D = \pm 1$

Th. 5 Eigenvalues of orthogonal matrix

Eigenvalues of orthogonal matrix are real or complex conjugates in pair with absolute value 1

Complex matrices, Hermitian, Skew-Hermitian, Unitary

Applications: numerous in quantum mechanics

Complex conjugate matrix: $\overline{\mathbf{A}} = \left[\overline{a}_{jk}\right]$

Conjugate transpose: $\overline{\mathbf{A}}^T = \left[\overline{a}_{kj}\right]$

Ex. 1

$$A = \begin{pmatrix} 3+4i & -5i \\ -7 & 6-2i \end{pmatrix} \Rightarrow \overline{A} = \begin{pmatrix} 3-4i & 5i \\ -7 & 6+2i \end{pmatrix} \Rightarrow \overline{A}^T = \begin{pmatrix} 3-4i & -7 \\ 5i & 6+2i \end{pmatrix}$$

Hermitian matrix: $\overline{\mathbf{A}}^T = \mathbf{A} \Rightarrow \left[\overline{a}_{kj}\right] = \left[a_{jk}\right]$

Skew-Hermitian matrix: $\overline{\mathbf{A}}^T = -\mathbf{A} \Rightarrow \left[\overline{a}_{kj}\right] = \left[-a_{jk}\right]$

Unitary matrix: $\overline{\mathbf{A}}^T = \mathbf{A}^{-1}$

Ex. 2 Hermitian, Skew-Hermitian and unitary

$$\begin{pmatrix} 4 & 1-3i \\ 1+3i & 7 \end{pmatrix} \qquad \begin{pmatrix} 3i & 2+i \\ -2+i & -i \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{pmatrix}$$

Consequences

If $\mathbf{A}$ Hermitian $\Rightarrow \overline{a}_{jj} = a_{jj} \Rightarrow$ real

If $\mathbf{A}$ Skew-Hermitian $\Rightarrow \overline{a}_{jj} = -a_{jj} \Rightarrow$ pure imaginary or zero

If a Hermitian matrix is real then $\overline{\mathbf{A}}^T = \mathbf{A}^T = \mathbf{A} \Rightarrow$ symmetric

If a skew-Hermitian matrix is real then $\overline{\mathbf{A}}^T = \mathbf{A}^T = -\mathbf{A} \Rightarrow$ skew-symmetric

If an unitary matrix is real then $\overline{\mathbf{A}}^T = \mathbf{A}^T = \mathbf{A}^{-1} \Rightarrow$ orthogonal

Hermitian, Skew-Hermitian and Unitary matrices generalize symmetric, skew-symmetric and orthogonal matrices respectively

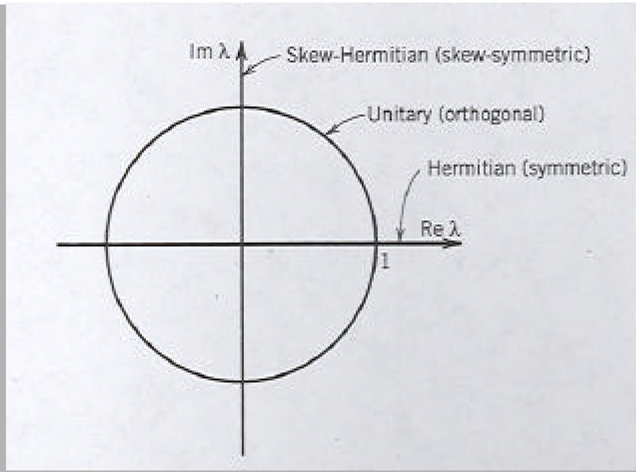
Eigenvalues

Th. 1 Eigenvalues

a) Eigenvalues of Hermitian matrix (symmetric matrix) are **real**

b) Eigenvalues of skew-Hermitian matrix (skew-symmetric) are **pure imaginary** or zero

c) Eigenvalues of unitary matrix (orthogonal matrix) have **absolute value 1**



Ex. 3

Hermitian: $\begin{pmatrix} 4 & 1-3i \\ 1+3i & 7 \end{pmatrix} \Rightarrow l^2 - 11l + 18 = 0 \Rightarrow \begin{matrix} l_1 = 9 \\ l_2 = 2 \end{matrix}$

Skew-Hermitian: $\begin{pmatrix} 3i & 2+i \\ -2+i & -i \end{pmatrix} \Rightarrow l^2 - 2il + 8 = 0 \Rightarrow \begin{matrix} l_1 = 4i \\ l_2 = -2i \end{matrix}$

Unitary: $\begin{pmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{pmatrix} \Rightarrow l^2 - il - 1 = 0 \Rightarrow \begin{matrix} l_1 = \frac{1}{2}\sqrt{3} + \frac{1}{2}i \\ l_2 = -\frac{1}{2}\sqrt{3} + \frac{1}{2}i \end{matrix}$

The absolute value of the last eigenvalues: $\left| \pm \frac{1}{2}\sqrt{3} + \frac{1}{2}i \right|^2 = \frac{3}{4} + \frac{1}{4} = 1$

Forms

$\bar{\mathbf{x}}^T \mathbf{A}\mathbf{x}$ = **form in the components** $x_1, \ldots x_n$ of $\mathbf{x}$ and $\mathbf{A}$ = **coefficient matrix**

For $n = 2$

$$\begin{bmatrix} \bar{x}_1 & \bar{x}_2 \end{bmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 \end{bmatrix} \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} =$$

$$= a_{11}\bar{x}_1 x_1 + a_{12}\bar{x}_1 x_2 + a_{21}\bar{x}_2 x_1 + a_{22}\bar{x}_2 x_2$$

In general:

$\bar{\mathbf{x}}^T \mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}\bar{x}_j x_k$ ➔ a sum of n^2 terms

For $\mathbf{x}$ and $\mathbf{A}$ real: $\bar{\mathbf{x}}^T \mathbf{A}\mathbf{x} = \mathbf{x}^T \mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} x_j x_k$ ➔ **quadratic form**

Coefficient matrix must be symmetric ➔ we can takeoff diagonal terms together in pairs and write the results as a sum of 2 equal terms

Ex. 4 $\bar{\mathbf{x}}^T\mathbf{A}\mathbf{x} = [x_1 \quad x_2]\begin{pmatrix} 3 & 4 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2 = 3x_1^2 + 10x_1x_2 + 2x_2^2$

Coefficient matrix can be rewritten as: $\begin{pmatrix} 3 & 5 \\ 5 & 2 \end{pmatrix}$ which is symmetric

If coefficient matrix Hermitian ➔ **Hermitian form**

If coefficient matrix Skew-Hermitian ➔ **Skew-Hermitian form**

Th. 2 Hermitian and Skew-Hermitian forms

For every choice of vector **x** the value of Hermitian form is real and value of Skew-Hermitian form is pure imaginary or zero

Ex. 5

$$\mathbf{x} = \begin{pmatrix} 1+i \\ 2i \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} 3 & 2-i \\ 2+i & 4 \end{pmatrix} \Rightarrow \bar{\mathbf{x}}^T\mathbf{A}\mathbf{x} = [1-i \quad -2i]\begin{pmatrix} 3(1+i)+(2-i)2i \\ (2+i)(1+i)+4\cdot 2i \end{pmatrix} = 34$$

Properties of Unitary matrices and complex vector space

For complex vector space C^n inner product $\mathbf{a}\cdot\mathbf{b}=\bar{\mathbf{a}}^T\mathbf{b}$

Length or **norm** $\|\mathbf{a}\|=\sqrt{\mathbf{a}\cdot\mathbf{a}}=\sqrt{\bar{\mathbf{a}}^T\mathbf{a}}=\sqrt{\bar{a}_1a_1+\ldots+\bar{a}_na_n}=\sqrt{|a_1|^2+\ldots+|a_n|^2}$

Note: for real vector ➔ reduces to usual norms

Th. 3 Invariance of inner product

A unitary transformation $\mathbf{y}=\mathbf{A}\mathbf{x}$ with a unitary matrix $\mathbf{A}$ preserves the value of the inner product hence of the norm

Complex analog of an orthonormal system of real vectors is a unitary system defined as

$$\mathbf{a}_j\cdot\mathbf{a}_k=\bar{\mathbf{a}}_j^T\mathbf{a}_k=\begin{cases}0 \text{ for j} \neq \text{k}\\ 1 \text{ for } j=k\end{cases}$$

Th. 4 Unitary systems of column and row vectors

A square matrix is unitary if and only if its column vectors (and row vectors) form a unitary system

Th. 5 Determinant of unitary matrix

The determinant of a unitary matrix has absolute value 1

Ex. 6

For $\mathbf{a}^T=[1 \quad i]$ and $\mathbf{b}^T=[3i \quad 2+i] \quad \Rightarrow \bar{\mathbf{a}}^T\mathbf{b}=3i-i(2+i)=1+i$

For $\mathbf{A}=\begin{pmatrix}0.6i & 0.8\\ 0.8 & 0.6i\end{pmatrix} \Rightarrow \mathbf{Aa}=\begin{pmatrix}1.4i\\ 0.2\end{pmatrix} \Rightarrow \mathbf{Ab}=\begin{pmatrix}-0.2+0.8i\\ -0.6+3.6i\end{pmatrix}$

$\Rightarrow \left(\bar{\mathbf{A}}\bar{\mathbf{a}}\right)^T\mathbf{Ab}=1+i$ the matrix $\mathbf{A}$ is unitary: $\bar{\mathbf{a}}_1^T\mathbf{a}_1=\bar{\mathbf{a}}_2^T\mathbf{a}_2=1$ and $\bar{\mathbf{a}}_1^T\mathbf{a}_2=0$

➔ determinant = -1

Similarity of matrices, basis of eigenvectors and diagonalization

Eigenvectors of $n \times n$ matrix **A** may (or may not) form a basis for R^n or C^n

If they do they can be used to diagonalize **A** ➔ eigenvalues on the main diagonal

Similarity of matrices

An $n \times n$ matrix $\hat{\mathbf{A}}$ is similar to **A** if $\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ (**similarity transformation**) for some nonsingular $n \times n$ matrix **P**

Th. 1 Eigenvalues and eigenvectors of similar matrices

If $\hat{\mathbf{A}}$ similar to **A** then the eigenvalues are the same

Furthermore if $\mathbf{x}$ is an eigenvector of **A** then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to same eigenvalues

Th. 2 Linear independence of eigenvectors

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of a $n \times n$ matrix. Then corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_k$ form a linearly independent set

Th. 3 Basis of eigenvectors

If a $n \times n$ matrix **A** has n distinct eigenvalues then **A** has a basis of eigenvectors for C^n or R^n

Actually, basis of eigenvectors exist under more general conditions than those given in Th. 3

Th. 4 Basis of eigenvectors

A Hermitian, Skew-Hermitian or Unitary matrix has a basis of eigenvectors for C^n that is a Unitary system

A symmetric matrix has an orthonormal basis of eigenvectors for R^n

A basis of eigenvectors of matrix **A** is of great advantage for transformation $\mathbf{y} = \mathbf{A}\mathbf{x}$

Writing in terms of basis $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n$

$$\Rightarrow \mathbf{y} = \mathbf{A}\mathbf{x} = \mathbf{A}\left(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n\right) = c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \ldots + c_n\mathbf{A}\mathbf{x}_n = c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \ldots + c_n\lambda_n\mathbf{x}_n$$

Decomposed action of $\mathbf{A}$ on $\mathbf{x}$ into simple multiplication by scalar (eigenvalues) on eigenvectors of $\mathbf{A}$

Diagonalization

Th. 5 Diagonalization of matrix

If a $n \times n$ matrix $\mathbf{A}$ has a basis of eigenvectors then $\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$ is diagonal with eigenvalues of $\mathbf{A}$ as the entries on main diagonal

$\mathbf{X}$ is a matrix with eigenvectors as column vectors

Also $\mathbf{D}^{\mathbf{m}} = \mathbf{X}^{\mathbf{1}}\mathbf{A}^{\mathbf{m}}\mathbf{X}$ $(m = 2, 3, \ldots)$

Ex. 7

$\mathbf{A} = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ ➔ eigenvectors: $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ hence $\mathbf{X} = \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}$

$$\Rightarrow \mathbf{X}^{-1}\mathbf{A}\mathbf{X} = -\frac{1}{5}\begin{pmatrix} -1 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0.2 & 0.2 \\ 0.2 & 0.8 \end{pmatrix}\begin{pmatrix} 24 & 1 \\ 6 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$$

Ex. 8

$$A = \begin{pmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{pmatrix}$$

Characteristic determinant = characteristic equation

$$-\lambda^3 - \lambda^2 + 12\lambda = 0 \Rightarrow \lambda_1 = 3,\ \lambda_2 = -4,\ \lambda_3 = 0$$

Eigenvalues + Gauss elimination to $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ ➔ eigenvectors

$$\begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \Rightarrow X = \begin{pmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{pmatrix}$$

Gauss-Jordan elimination ➔ $\mathbf{X}^{-1} = \begin{pmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{pmatrix}$

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Transformation of forms to principal axis

Quadratic form: $Q = \mathbf{x}^T A\mathbf{x}$

Since **A** symmetric ➔ has orthonormal basis of eigenvectors

$\Rightarrow \mathbf{X}^{-1} = \mathbf{X}^T \Rightarrow \mathbf{A} = \mathbf{XDX}^{-1} = \mathbf{XDX}^T$

$\Rightarrow Q = \mathbf{x}^T\mathbf{XDX}^T\mathbf{x}$

Putting $\mathbf{X}^T\mathbf{x} = \mathbf{y} \Rightarrow \mathbf{XX}^T\mathbf{x} = \mathbf{Xy}$

Furthermore: $(\mathbf{x}^T\mathbf{X}) = (\mathbf{X}^T\mathbf{x})^T = \mathbf{y}^T$

$\Rightarrow Q = \mathbf{y}^T\mathbf{Dy} = l_1 y_1^2 + l_2 y_2^2 + \ldots + l_n y_n^2$

Th. 6 principal axes theorem

$$Q = \mathbf{x}^T\mathbf{Ax} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} x_j x_k$$

Substitution $\mathbf{x} = \mathbf{Xy}$ transform quadratic form into principal axes form where $l_1, l_2, \ldots, l_n$ are the eigenvalues of the symmetric matrix **A** and **X** is an orthogonal matrix with corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ as column vectors

Ex. 9

$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$

$\Rightarrow Q = \mathbf{x}^T \mathbf{A}\mathbf{x}$ where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $A = \begin{pmatrix} 17 & -15 \\ -15 & 17 \end{pmatrix}$

Characteristic equation: $(17 - l)^2 - 15^2 = 0 \Rightarrow l_1 = 2$ and $l_2 = 32$

$\Rightarrow Q = 2y_1^2 + 32y_2^2 = 128$

Ellipse: $\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$

To find the direction of principal axis ➔ normalized eigenvectors

Solving $(\mathbf{A} - l\,\mathbf{I})\,\mathbf{x} = 0$ with two eigenvalues ➔ $\mathbf{x}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

Hence $\mathbf{x} = \mathbf{X}\mathbf{y} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ this is a rotation by 45°