

Systems of ODE

Example of system with 2 ODE: $y_1' = a_{11}y_1 + a_{12}y_2$
 $y_2' = a_{21}y_1 + a_{22}y_2$

$$y_1' = a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n$$

In general for a system of n ODE $y_2' = a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n$

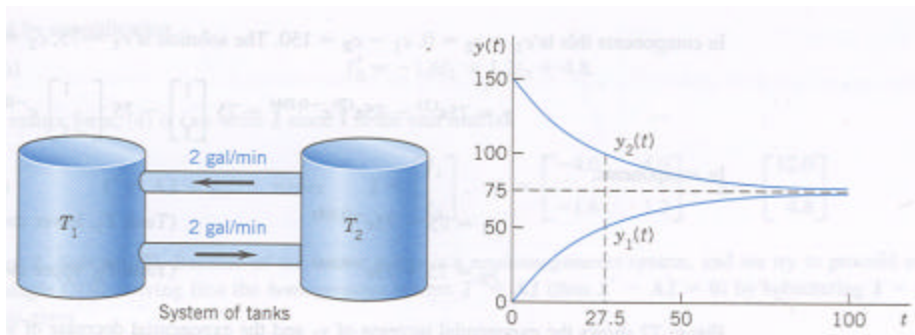
$$y_n' = a_{n1}y_1 + a_{n2}y_2 + \dots + a_{nn}y_n$$

Differentiation of matrix: $\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \Rightarrow \mathbf{y}'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix}$

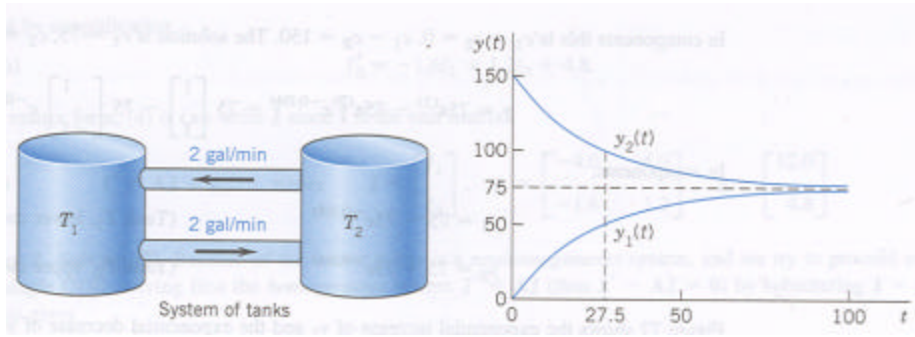
Usually can take the form $\mathbf{y}'(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \mathbf{A}\mathbf{y} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Or in general $\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$

Example 1: mixing problem with 2 tanks



- Each contain 100 gal of water
- In T_1 we have pure water and in T_2 we have 150 lb of fertilizer
- Circulation of water 2gal/min (and stirring \rightarrow instantaneous mixing)
- Concentration of fertilizer with time $y_1(t)$ and $y_2(t)$



Model: $y'(t) = \frac{\text{inflow}}{\text{min}} - \frac{\text{outflow}}{\text{min}}$

This gives 2 ODEs: $y_1' = \frac{2}{100}y_2 - \frac{2}{100}y_1$

$$y_2' = \frac{2}{100}y_1 - \frac{2}{100}y_2$$

Equivalent to $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

We try a solution of the form: $\mathbf{y} = \mathbf{x}e^{I t} \Rightarrow \mathbf{y}' = \mathbf{I} \mathbf{x} e^{I t} = \mathbf{A} \mathbf{x} e^{I t}$

This is an eigenvalues problem: $\mathbf{A} \mathbf{x} = \mathbf{I} \mathbf{x} \Rightarrow \mathbf{A} \mathbf{x} - \mathbf{I} \mathbf{x} = \mathbf{0}$

With nontrivial solution if $\text{Det}(\mathbf{A} - \mathbf{I} \mathbf{I}) = 0$

$$\Rightarrow \begin{vmatrix} -0.02 - \mathbf{I} & 0.02 \\ 0.02 & -0.02 - \mathbf{I} \end{vmatrix} = (-0.02 - \mathbf{I})^2 - (0.02)^2 = \mathbf{I}(\mathbf{I} + 0.04) = 0$$

Two eigenvalues $\mathbf{I}_1 = 0$ and $\mathbf{I}_2 = -0.04$ and two eigenvectors $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

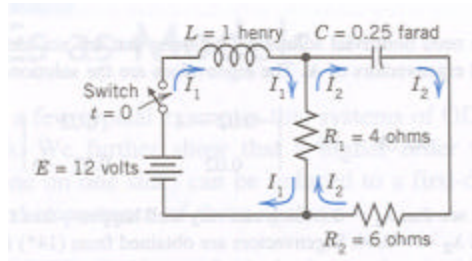
The solution $\mathbf{y} = c_1 \mathbf{x}^{(1)} e^{I t} + c_2 \mathbf{x}^{(2)} e^{I t} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-0.04 t}$ where c_1 and c_2 are arbitrary constants

Initial conditions: $y_1(0) = 0$ and $y_2(0) = 150$

$$\Rightarrow \mathbf{y}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 150 \end{pmatrix} \Rightarrow \begin{cases} c_1 + c_2 = 0 \\ c_1 - c_2 = 150 \end{cases} \Rightarrow c_1 = 75 \text{ and } c_2 = -75$$

At what time t_1 concentration is $\frac{1}{2}$ that of t_2 ? This happens when t_1 contains $\frac{1}{3}$ of total amount $\Rightarrow y_1 = 75 - 75e^{-0.04 t} = 50 \Rightarrow e^{-0.04 t} = \frac{1}{3} \Rightarrow t = \frac{\ln(3)}{0.04} = 27.5 \text{ min}$

Example 2 Electric network



We want to know the currents $I_1(t)$ and $I_2(t)$ in the two loops of the above circuit and we assume all currents and charges to be zero at $t = 0$

The model is based on **Kirchhoff's voltage law**:

In the left loop

- the voltage drops over the inductor $LI_1' = I_1' [V]$
- the voltage drops over the resistor $R_1(I_1 - I_2) = 4(I_1 - I_2) [V]$
- The sum of voltage must be equal to that of the battery $I_1' + 4(I_1 - I_2) = 12$

In the right loop

- the voltage drops over the first resistor $R_1(I_2 - I_1) = 4(I_2 - I_1) [V]$
- the voltage drops over the resistor $R_2 I_2 = 6I_2 [V]$
- the voltage drops over the capacitor $\frac{1}{C} \int I_2 dt = 4 \int I_2 dt [V]$
- The sum must be equal to zero
 $6I_2 + 4(I_2 - I_1) + 4 \int I_2 dt = 0$ or $10I_2 - 4I_1 + 4 \int I_2 dt = 0$

Dividing by ten and deriving $I_2' - 0.4I_1' + 0.4I_2 = 0$

Replacing by the value of I_1' in left loop: $I_2' - 0.4(12 - 4(I_1 - I_2)) + 0.4I_2 = 0$

$$\Rightarrow I_2' = -1.6I_1 + 1.2I_2 + 4.8$$

In matrix form $\mathbf{J}' = \mathbf{A}\mathbf{J} + \mathbf{g} = \begin{pmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} 12.0 \\ 4.8 \end{pmatrix}$

This is a nonhomogeneous system

We first look for a solution of the homogenous system $\mathbf{J}' = A\mathbf{J} = \begin{pmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$

trying $\mathbf{J} = \mathbf{x}e^{I t} \Rightarrow \mathbf{J}' = I \mathbf{x}e^{I t} = A\mathbf{x}e^{I t}$

The nontrivial solution $\det(A - I I) = 0 \Rightarrow \begin{vmatrix} (-4.0 - I) & 4.0 \\ -1.6 & 1.2 - I \end{vmatrix} = 0$
 $\Rightarrow I^2 + 2.8I + 1.6 = 0$ with two eigenvalues $I_1 = -2$ and $I_2 = -0.8$

And two eigenvectors $\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0.8 \end{pmatrix}$

Consistent with general homogenous solution

$$\mathbf{J}_h = c_1 \mathbf{x}^{(1)} e^{-2t} + c_2 \mathbf{x}^{(2)} e^{-0.8t} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{-0.8t}$$

To find a particular solution, we note that since \mathbf{g} is a constant we can try a constant

$$\Rightarrow \mathbf{J}_p = \mathbf{a} \text{ such that } A\mathbf{a} + \mathbf{g} = 0 \Rightarrow \begin{pmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 12.0 \\ 4.8 \end{pmatrix} = 0$$

$$\begin{aligned} -4.0a_1 + 4.0a_2 + 12.0 &= 0 \\ -1.6a_1 + 1.2a_2 + 4.8 &= 0 \end{aligned} \Rightarrow a_1 = 3 \text{ and } a_2 = 0 \text{ or } \mathbf{a} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

So the general solution

$$\mathbf{J} = \mathbf{J}_h + \mathbf{J}_p = c_1 \mathbf{x}^{(1)} e^{-2t} + c_2 \mathbf{x}^{(2)} e^{-0.8t} + \mathbf{a} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{-0.8t} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$I_1 = 2c_1 e^{-2t} + c_2 e^{-0.8t} + 3 \text{ and } I_2 = c_1 e^{-2t} + 0.8c_2 e^{-0.8t}$$

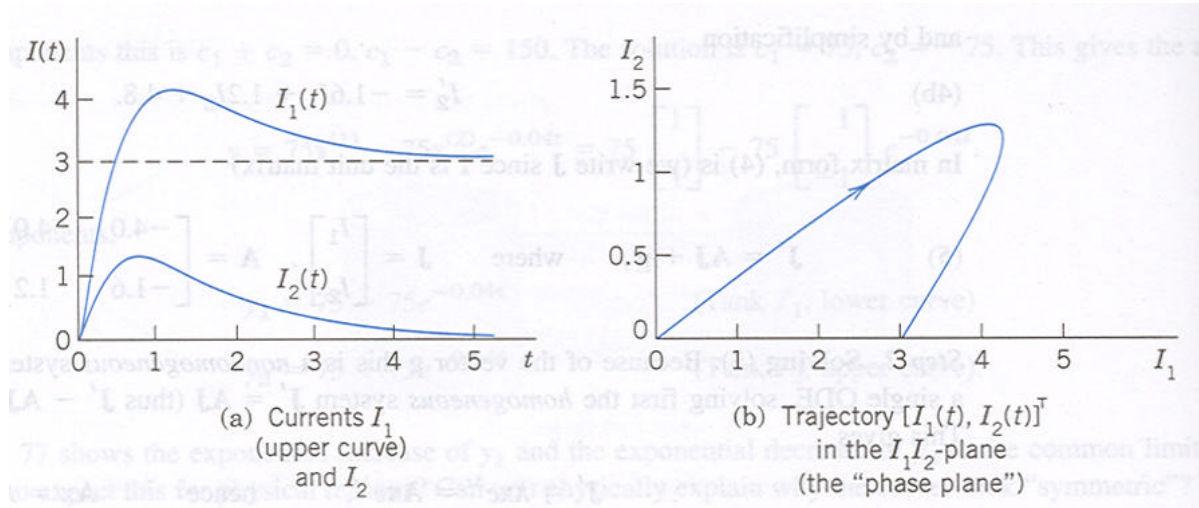
Applying the initial conditions

$$I_1 = 2c_1 + c_2 + 3 = 0 \text{ and } I_2 = c_1 + 0.8c_2 = 0 \Rightarrow c_1 = -4 \text{ and } c_2 = 5$$

Solution

$$\mathbf{J} = \mathbf{J}_h + \mathbf{J}_p = -4.0\mathbf{x}^{(1)}e^{-2t} + 5.0\mathbf{x}^{(2)}e^{-0.8t} + \mathbf{a} = -4.0\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-2t} + 5.0\begin{pmatrix} 1 \\ 0.8 \end{pmatrix}e^{-0.8t} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$I_1 = -8.0e^{-2t} + 5.0e^{-0.8t} + 3 \text{ and } I_2 = -4.0e^{-2t} + 4.0e^{-0.8t}$$



Solutions shown individually look like two damp vibrations

Phase plane

Shows the solution using I_1 and I_2 as degrees of liberty of system (parametric representation) where the arrow shows the sense of variation with time

- The current I_2 rises almost linearly in time with I_1 both reaching a maximum
- But then I_2 drops to zero while the current I_1 reach a lower constant value below its maximum

Conversion of nth-order ODE to a system

- Permits study and solution of single ODE by method of system
- Open way of including theory of higher order ODE into that of first order system

Nth-order ODE

$$y^{(n)} = F(t, y, y', \dots, y^{n-1})$$

Transformations: $y_1 = y$, $y_2 = y'$, $y_3 = y''$, ..., $y_n = y^{n-1}$

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \end{aligned}$$

Such that $y_n' = F(t, y_1, y_2, \dots, y_n)$

And consequently $y^{(n)} = y_n'$

Example 3: mass on a spring

$$my'' + cy' + ky = 0 \text{ or } y'' = -\frac{c}{m}y' - \frac{k}{m}y$$

Applying the transformations: $y_1 = y$, $y_2 = y'$, $y_3 = y''$

$$\Rightarrow y_1' = y_2 \text{ and } y_2' = y_3 = y'' = -\frac{c}{m}y' - \frac{k}{m}y$$

$$\text{Setting } \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \mathbf{y}' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \mathbf{A}\mathbf{y} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\text{The characteristic equation } \det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m}$$

Numerical example: $m = 1$, $c = 2$ and $k = 0.75$

$$\Rightarrow \mathbf{I}^2 + \frac{c}{m}\mathbf{I} + \frac{k}{m} = \mathbf{I}^2 + 2\mathbf{I} + 0.75 = (\mathbf{I} + 0.5)(\mathbf{I} + 1.5)$$

Two eigenvalues $\lambda_1 = -0.5$ and $\lambda_2 = -1.5$

And two eigenvectors $\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1.5 \end{pmatrix}$

Yields general solution $\mathbf{y} = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-0.5t} + c_2 \begin{pmatrix} 1 \\ -1.5 \end{pmatrix} e^{-1.5t}$

Therefore $y_1 = 2c_1 e^{-0.5t} + c_2 e^{-1.5t}$ and $y_2 = y_1' = -c_1 e^{-0.5t} - 1.5c_2 e^{-1.5t}$

Basic theory of system ODEs

General system

$$y_1' = f_1(t, y_1, y_2, \dots, y_n)$$

$$y_2' = f_2(t, y_1, y_2, \dots, y_n)$$

\vdots

$$y_n' = f_n(t, y_1, y_2, \dots, y_n)$$

We can write the system as $\mathbf{y}' = \mathbf{f}(t)$

A solution on some interval $a < t < b$ is a set of differentiable functions

$$y_1 = h_1(t), \dots, y_n = h_n(t)$$

In vector form $\mathbf{h} = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$ such that $\mathbf{y} = \mathbf{h}(t)$

Initial value problem \rightarrow n-initial conditions $y_1(t_0) = K_1, y_2(t_0) = K_2, \dots, y_n(t_0) = K_n$

In vector form $\mathbf{y}(t_0) = \mathbf{K}$

Theorem 1: existence and uniqueness of solution

Let f_1, \dots, f_n in $\mathbf{y}' = \mathbf{f}(t)$ be continuous functions having continuous partial derivatives

$\frac{df_1}{dy_1}, \dots, \frac{df_n}{dy_1}, \dots, \frac{df_n}{dy_n}$ in some domain R of t, y_1, y_2, \dots, y_n -space containing the point

(t_0, K_1, \dots, K_n) , then $\mathbf{y}' = \mathbf{f}(t)$ has a solution on some interval $t_0 - \mathbf{a} < t < t_0 + \mathbf{a}$, satisfying the initial value problem and this solution is unique

Linear system

Has the form $\mathbf{y}' = A\mathbf{y} + \mathbf{g}$ which is **nonhomogeneous** or **homogenous** when $\mathbf{g} = 0$

For a linear system we have $\frac{\partial f_1}{\partial y_1} = a_{11}(t), \dots, \frac{\partial f_n}{\partial y_n} = a_{nn}(t)$ in theorem 1

Theorem 2: existence and uniqueness of solution

Let a_{jk} and g_j be continuous functions on open interval $\mathbf{a} < t < \mathbf{b}$ containing the point

$t = t_0$, then $\mathbf{y}' = A\mathbf{y} + \mathbf{g}$ has a solution $\mathbf{y}(t)$ which is unique

Superposition principle

If $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ are solutions of the homogeneous linear system $\mathbf{y}' = A\mathbf{y}$ then the combination $\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)}$ is also a solution

PROOF: We simply take the derivative

$$\mathbf{y}' = c_1\mathbf{y}^{(1)'} + c_2\mathbf{y}^{(2)'} = c_1A\mathbf{y}^{(1)} + c_2A\mathbf{y}^{(2)} = A(c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)}) = A\mathbf{y}$$

Basis, General solution and Wronskian

A **basis** or **fundamental system** of solutions of homogeneous ODEs $\mathbf{y}' = A\mathbf{y}$ in some interval J is a linearly independent set of n -solutions $\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(n)}$ of homogeneous ODEs and the general solution is a combination $\mathbf{y} = c_1\mathbf{y}^{(1)} + \dots + c_n\mathbf{y}^{(n)}$ where c_1, \dots, c_n are arbitrary constants

We can write the n -solutions $\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(n)}$ as column of $n \times n$ matrix $Y = \begin{bmatrix} \mathbf{y}^{(1)} & \dots & \mathbf{y}^{(n)} \end{bmatrix}$

And the determinant of this matrix is **Wronskian**

$$W(\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(n)}) = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} & \dots & y_1^{(n)} \\ y_2^{(1)} & y_2^{(2)} & \dots & y_2^{(n)} \\ \cdot & \cdot & \dots & \cdot \\ y_n^{(1)} & y_n^{(2)} & \dots & y_n^{(n)} \end{vmatrix}$$

The solution form a basis on J if and only if $W \neq 0$ at any time t_1 in this interval

Then $Y = \begin{bmatrix} \mathbf{y}^{(1)} & \dots & \mathbf{y}^{(n)} \end{bmatrix}$ is a **fundamental matrix**

$$\text{Then } \mathbf{y} = Y\mathbf{c} \text{ where } \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

Particular case: if y and z are solutions of 2-order homogeneous ODE then

$$W(y, z) = \begin{vmatrix} y & z \\ y' & z' \end{vmatrix}$$

Constant coefficient systems

Assume $\mathbf{y}' = A\mathbf{y}$

With constant coefficients $\Rightarrow n \times n$ matrix $A = [a_{jk}]$ all independent of t

Solution $\mathbf{y}' = K\mathbf{y}$ using $y = ce^{kt} \Rightarrow \mathbf{y} = \mathbf{x}e^{I t} \Rightarrow \mathbf{y}' = I\mathbf{x}e^{I t} = A\mathbf{y}$

Eigenvalues problem $A\mathbf{x} - I\mathbf{x} = 0$

If A has linearly independent set of eigenvalues, example A symmetric ($a_{kj} = a_{jk}$) or skew symmetric ($a_{kj} = -a_{jk}$) or n -different eigenvalues I_1, \dots, I_n then we have

$$\begin{aligned} \mathbf{y}^{(1)} &= \mathbf{x}^{(1)} e^{I_1 t} \\ \mathbf{y}^{(2)} &= \mathbf{x}^{(2)} e^{I_2 t} \\ &\vdots \\ \mathbf{y}^{(n)} &= \mathbf{x}^{(n)} e^{I_n t} \end{aligned}$$

$\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ eigenvectors and a related set of solutions

The Wronskian

$$W(\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(n)}) = \begin{vmatrix} x_1^{(1)} e^{I_1 t} & x_1^{(2)} e^{I_2 t} & \dots & x_1^{(n)} e^{I_n t} \\ x_2^{(1)} e^{I_1 t} & x_2^{(2)} e^{I_2 t} & \dots & x_2^{(n)} e^{I_n t} \\ \cdot & \cdot & \dots & \cdot \\ x_n^{(1)} e^{I_1 t} & x_n^{(2)} e^{I_2 t} & \dots & x_n^{(n)} e^{I_n t} \end{vmatrix} = e^{I_1 t + I_2 t + \dots + I_n t} \begin{vmatrix} x_1^{(1)} & x_1^{(2)} & \dots & x_1^{(n)} \\ x_2^{(1)} & x_2^{(2)} & \dots & x_2^{(n)} \\ \cdot & \cdot & \dots & \cdot \\ x_n^{(1)} & x_n^{(2)} & \dots & x_n^{(n)} \end{vmatrix}$$

- The first term is an exponential which is never zero
- The second term is not zero because columns are independent eigenvectors

General solution

If constant matrix in $\mathbf{y}' = A\mathbf{y}$ has linearly independent set of eigenvalues then

$\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(n)}$ form a basis and the general solution is $\mathbf{y} = c_1 \mathbf{x}^{(1)} e^{I_1 t} + c_2 \mathbf{x}^{(2)} e^{I_2 t} + \dots + c_n \mathbf{x}^{(n)} e^{I_n t}$

Phase plane

Qualitative method of obtaining general qualitative information on solutions without actually solving ODE or system

Created by **Henri Poincaré** (1854-1912) French mathematician who worked on complex analysis, divergent series, topology and astronomy

In physics – phase plane is $r-mr' = r-mv = r-p$

Consider two ODEs then $\mathbf{y}' = A\mathbf{y}$ takes the form

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

With solution $\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$

which can be graphed as two simple curves, or together in the phase plane y_1, y_2

The parametric curve in this phase plane is the **trajectory**

Example 4 $\mathbf{y}' = A\mathbf{y} \Rightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

The characteristic equation $\det(A - \lambda I) = \lambda^2 + 6\lambda + 8 = (\lambda + 2)(\lambda + 4) = 0$

Two eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -4$

corresponding to two eigenvectors

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

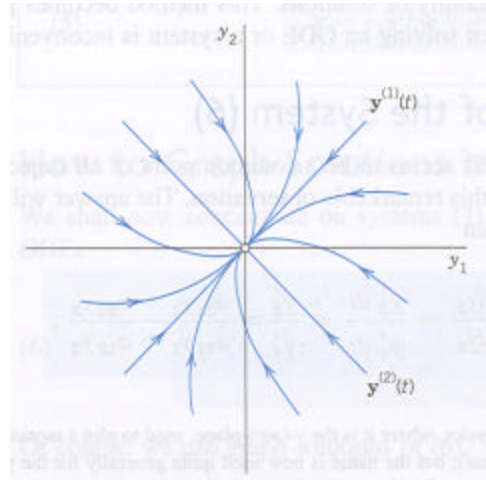
and the general solution

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$$

Phase portrait – two straight trajectories

$$c_1 = 0 \text{ and } c_2 = 0$$

Other values of constants yields other trajectories



Critical points

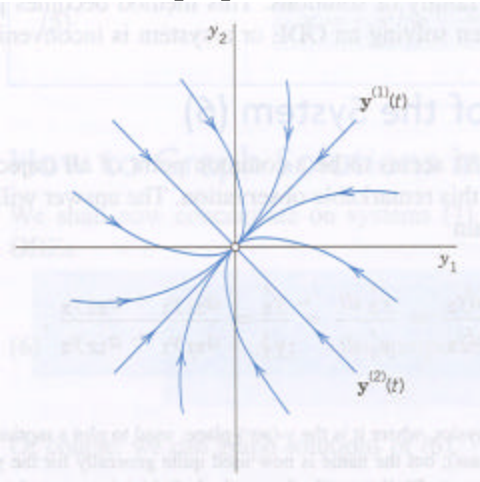
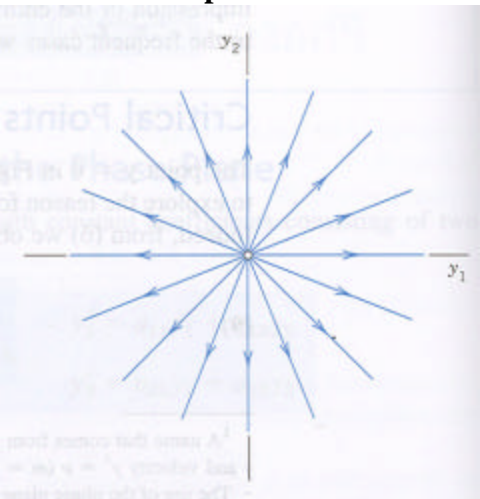
In example 4, the point 0 is a **critical point** = common point for all the trajectories

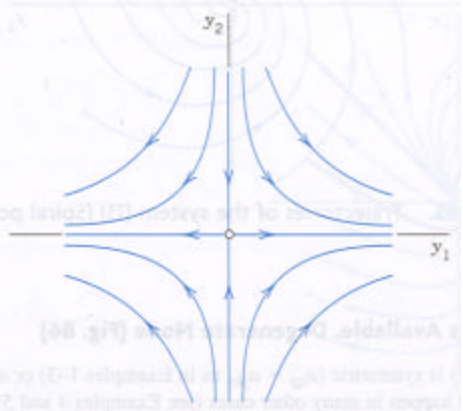
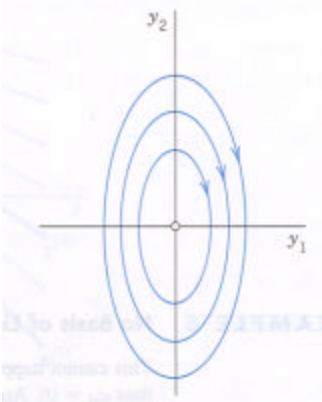
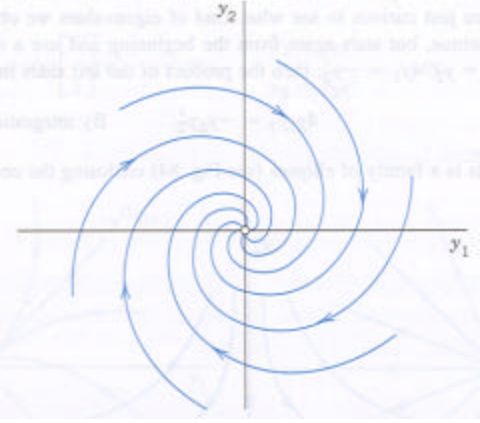
From $\mathbf{y}' = \mathbf{A}\mathbf{y}$ we get that $\frac{dy_2}{dy_1} = \frac{y_2' dt}{y_1' dt} = \frac{y_2'}{y_1'} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}$

This associates with every point $P(y_1, y_2)$ a unique tangent direction $\frac{dy_2}{dy_1}$ of trajectory

passing through P , except at $P_0 = (0,0)$ where $\frac{dy_2}{dy_1} = \frac{0}{0}$ which is a **critical point**

There are 5 different types of **critical points**:

<p style="text-align: center;">Improper node</p> 	<p><i>All the trajectories except two have the same limiting direction of the tangent</i></p> <p><i>The two exceptional trajectories also have limiting direction of tangent at P_0 but they are different</i></p> <p>The common limiting direction at zero is that of $\mathbf{x}^{(1)} = [1 \ 1]^T$ because e^{-4t} goes faster to zero than e^{-2t}</p> <p>The two exceptional directions are $\mathbf{x}^{(2)} = [1 \ -1]^T$ and $-\mathbf{x}^{(2)} = [-1 \ 1]^T$</p>
<p style="text-align: center;">Proper node</p> 	<p><i>Every trajectory as limiting direction and for any given direction \mathbf{d} at P_0 there is a trajectory having \mathbf{d} as its limiting direction</i></p> <p>Unit matrix $\mathbf{y}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$</p> <p>Characteristic equation $(1 - I)^2 = 0$</p> <p>Solution $\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$</p> <p>Trajectories $c_1 y_2 = c_2 y_1$ (linear)</p>

<p style="text-align: center;">Saddle point</p>  <p style="text-align: center;">family of hyperbolas + coordinate axes</p>	<p>Two incoming and two outgoing trajectories through P_0 and all the other trajectories in neighborhood of P_0 bypass P_0</p> <p>System $\mathbf{y}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$</p> <p>Characteristic equation $(\mathbf{I} - \mathbf{I})(-\mathbf{I} - \mathbf{I}) = 0$</p> <p>Solution $\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$</p> <p>Trajectories $y_1 y_2 = \text{constant}$ (hyperbolas)</p>
<p style="text-align: center;">Center</p>  <p style="text-align: center;">$2y_1^2 + \frac{1}{2}y_2^2 = \text{constant}$ family of ellipses</p>	<p>Critical point is enclosed by infinitely many trajectories</p> <p>System $\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$</p> <p>Characteristic equation $\mathbf{I}^2 + 4 = 0$</p> <p>with solution $\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2it} + c_2 \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{-2it}$</p> <p>Next step = transformation to real form by Euler formula</p> <p>Shortcut: multiply equations $y_1' = y_2$ and $y_2' = -4y_1 \Rightarrow 4y_1 y_1' = -y_2 y_2'$ and integrate</p>
<p style="text-align: center;">Spiral</p>  <p style="text-align: center;">$r = ce^{-t}$ family of spirals for each real c</p>	<p>Critical point P_0 about which the trajectories spiral approaching it as $t \rightarrow \infty$</p> <p>System $\mathbf{y}' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$</p> <p>Characteristic equation $\mathbf{I}^2 + 4 = 0$</p> <p>Solution $\mathbf{y} = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1+i)t} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1-i)t}$</p> <p>Shortcut: $y_1 y_1' + y_2 y_2' = -(y_1^2 + y_2^2) = -r^2$ in polar coordinates – differentiating $rr' = -r^2 \Rightarrow \frac{dr}{r} = -dt \Rightarrow \ln r = -t + c^*$</p>

Degenerated node → no basis

Example $\mathbf{y}' = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

The matrix A is **not skew symmetric** ($A^T \neq -A$)

Characteristic equation $(I - 3)^2 = 0 \Rightarrow$ double root $I = 3$

With corresponding one eigenvector $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and some non zero multiple of another

In fact $\mathbf{y}^{(2)} = \mathbf{x}te^{I t} + \boldsymbol{\mu}e^{I t}$ with $\boldsymbol{\mu} = [\mathbf{m}_1 \quad \mathbf{m}_2]^T$

Taking derivative $\mathbf{y}^{(2)'} = \mathbf{x}e^{I t} + \mathbf{x}Ite^{I t} + I\boldsymbol{\mu}e^{I t} = A\mathbf{y}^{(2)} = A\mathbf{x}e^{I t} + A\boldsymbol{\mu}e^{I t}$

Since $A\mathbf{x} = I\mathbf{x}$ the term $\mathbf{x}Ite^{I t}$ cancel then dividing by $e^{I t}$

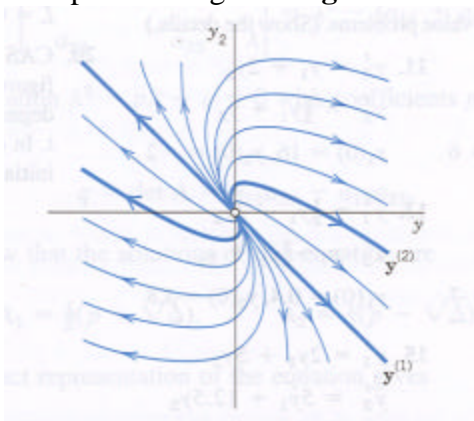
$$\mathbf{x} + I\boldsymbol{\mu} = A\boldsymbol{\mu} \Rightarrow (A - II)\boldsymbol{\mu} = \mathbf{x}$$

Where $I = 3$ and $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ so that $(A - 3I)\boldsymbol{\mu} = \begin{pmatrix} 4-3 & 1 \\ -1 & 2-3 \end{pmatrix} \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Thus we have $\mathbf{m}_1 + \mathbf{m}_2 = 1$ and $-\mathbf{m}_1 - \mathbf{m}_2 = -1$ which suggest $\boldsymbol{\mu} = (0 \quad 1)^T$

And the solution $\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + c_2 \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) e^{3t}$

Critical point at origin is **degenerate node**



$c_1\mathbf{y}^{(1)}$ gives heavy straight line – lower part $c_1 > 0$ and upper part $c_1 < 0$

$\mathbf{y}^{(2)}$ right part of heavy curve from 0 through second, first and fourth quadrant

$-\mathbf{y}^{(2)}$ gives the other part from 0 through fourth, third and second quadrant

Critical points vs. stability of solutions

Consider an system of homogeneous ODEs with constant coefficients

$$\mathbf{y}' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \mathbf{A}\mathbf{y} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

On the phase plane, the solution $\mathbf{y}(t) = (y_1(t) \ y_2(t))^T$ yields **trajectories = phase portrait**

With critical point such that $\frac{dy_2}{dy_1} = \frac{y_2' dt}{y_1' dt} = \frac{y_2'}{y_1'} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}$

The relation between critical point and eigenvalues pass by characteristic equation

$$\det(\mathbf{A} - \mathbf{I}\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + \det(\mathbf{A}) = 0$$

This is a quadratic equation $\lambda^2 - p\lambda + q = 0$ with solutions

$$\lambda_1 = \frac{1}{2}(p + \sqrt{\Delta}) \quad \text{and} \quad \lambda_2 = \frac{1}{2}(p - \sqrt{\Delta})$$

Where

- $p = (a_{11} + a_{22})$ is the **trace** of \mathbf{A}
- $q = \det(\mathbf{A}) = (a_{11}a_{22} - a_{12}a_{21})$
- $\Delta = p^2 - 4q$

If we express the solution in terms of product $\lambda^2 - p\lambda + q = (\lambda - \lambda_1)(\lambda - \lambda_2)$

Then we have

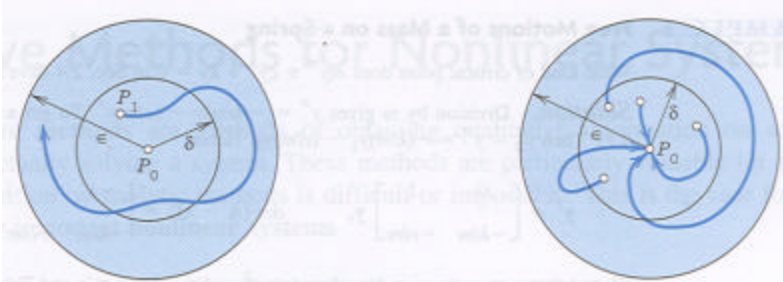
- $p = \lambda_1 + \lambda_2$
- $q = \lambda_1\lambda_2$
- $\Delta = (\lambda_1 - \lambda_2)^2$

Eigenvalue criteria for critical points

	$p = \lambda_1 + \lambda_2$	$q = \lambda_1\lambda_2$	$\Delta = (\lambda_1 - \lambda_2)^2$	eigenvalues
Node		$q > 0$	$\Delta \geq 0$	Real same sign
Saddle point		$q < 0$		Real opposite sign
Center	$p = 0$	$q > 0$		Pure imaginary
Spiral point	$p \neq 0$		$\Delta < 0$	Complex, not pure imaginary

Stability of critical points

- **Stable** all the trajectories close to P_0 remain close to P_0 in future

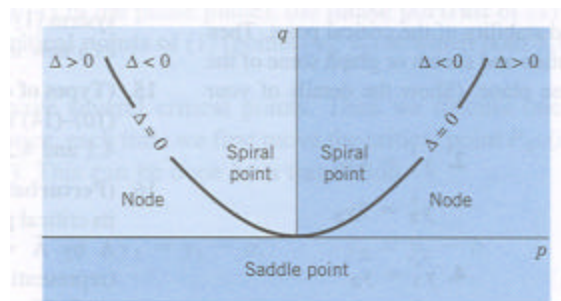


For every disk D_ϵ of radius $\epsilon > 0$ with center P_0 , there is a disk D_δ with radius $\delta > 0$ and center P_0 such that every trajectory that has point $P_1(t_1)$ in D_δ has all its subsequent points $t \geq t_1$ in D_ϵ

If condition not verified = **unstable**

- **Stable and attractive** - P_0 is stable and all trajectories with point within D_ϵ approaches P_0 as $t \rightarrow \infty$ (asymptotically stable)

Type of Stability	$p = \lambda_1 + \lambda_2$	$q = \lambda_1 \lambda_2$
(a) Stable and attractive	$p < 0$	$q > 0$
(b) Stable	$p \leq 0$	$q > 0$
(c) Unstable	$p > 0$	OR $q < 0$



For example, $q > 0$ implies $I_1 I_2$ have same sign or complex conjugate

In this case, $p < 0$ means eigenvalues are negative or have negative real part

If $\Delta < 0 \Rightarrow I_1 = a + ib$ and $I_2 = a - ib$ thus $p = 2a < 0$ implies **spiral attractive**

If $p > 0 \Rightarrow$ **unstable spiral point**

If $p = 0 \Rightarrow I_1 = -I_2 \Rightarrow q = -I_1^2$, but if $q > 0 \Rightarrow I_1^2 = -q < 0$ which implies that the two eigenvalues are pure imaginary \rightarrow **periodic** solutions with trajectories **closed curves around critical point**

Example 5 $y' = Ay \Rightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$p = -6, q = 8, \Delta = 4 \rightarrow$ node which is stable and attractive

Example 6 mass on spring

$$my'' + cy' + ky = 0 \text{ or } y'' = -\frac{c}{m}y' - \frac{k}{m}y$$

$$y' = Ay = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{vmatrix} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

$$p = -\frac{c}{m}, q = \frac{k}{m} \text{ and } \Delta = \left(\frac{c}{m}\right)^2 - 4\frac{k}{m}$$

No damping $\Rightarrow c = 0$ which yields $p = 0$ and $q > 0$ the critical point is a **center**

Underdamping $c^2 < 4mk$ which yields $p < 0, q > 0, \Delta < 0$ **stable attractive spiral**

Critical damping $c^2 = 4mk$ which yields $p < 0, q > 0, \Delta = 0$ **stable attractive node**

Overdamping $c^2 > 4mk$ which yields $p < 0, q > 0, \Delta > 0$ **stable attractive node**

Numerical example: $m = 1$, $c = 2$ and $k = 0.75$

$$\Rightarrow \mathbf{I}^2 + \frac{c}{m}\mathbf{I} + \frac{k}{m} = \mathbf{I}^2 + 2\mathbf{I} + 0.75 = (\mathbf{I} + 0.5)(\mathbf{I} + 1.5)$$

Two eigenvalues $\lambda_1 = -0.5$ and $\lambda_2 = -1.5$

And two eigenvectors $\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1.5 \end{pmatrix}$

Yields general solution $\mathbf{y} = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-0.5t} + c_2 \begin{pmatrix} 1 \\ -1.5 \end{pmatrix} e^{-1.5t}$

Therefore $y_1 = 2c_1 e^{-0.5t} + c_2 e^{-1.5t}$ and $y_2 = y_1' = -c_1 e^{-0.5t} - 1.5c_2 e^{-1.5t}$

Non linear systems

Solution by analytic method is difficult or impossible

First order non linear systems: $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ and thus $y_1' = f_1(y_1, y_2)$
 $y_2' = f_2(y_1, y_2)$

If system is autonomous (does not depend on t) **extended phase plane methods** give characterization of various general properties of solutions

- Advantage over numerical methods, which give only one (approximate) solution at a time (but with higher precision)

If first order nonlinear solution has several critical points

- discuss one after the other each time moving point $P_0 : (a, b)$ to be discussed at origin \Rightarrow applying translation $\tilde{y}_1 = y_1 - a$ and $\tilde{y}_2 = y_2 - b$
- P_0 is isolated – only critical point within sufficiently small circular disk with center on origin

Linearization of nonlinear systems

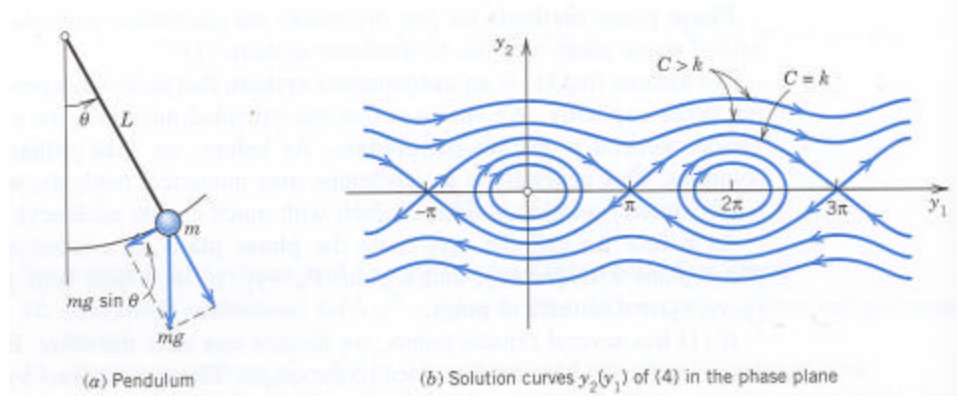
Assume f_1 and f_2 are continuous and have continuous partial derivatives in neighborhood of P_0

- A linear system approximating $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ have same kind of critical points (with two exceptions)
- Since P_0 is critical point $f_1(0,0) = f_2(0,0) = 0$
- Transformation $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{h}(\mathbf{y})$ and thus $y_1' = a_{11}y_1 + a_{12}y_2 + h_1(y_1, y_2)$
 $y_2' = a_{21}y_1 + a_{22}y_2 + h_2(y_1, y_2)$
- If $\text{Det } \mathbf{A} \neq 0$ then the type and stability of P_0 is the same as that of critical point of linear system
- $\mathbf{y}' = \mathbf{A}\mathbf{y}$ and thus $y_1' = a_{11}y_1 + a_{12}y_2$
 $y_2' = a_{21}y_1 + a_{22}y_2$

The above assumption on derivatives implies that h_1 and h_2 are small near P_0

Two exceptions: *eigenvalues are equal (degenerated) or pure imaginary, then in addition to the type of critical points of linear system the nonlinear system may have a spiral point*

Example 1 – free undamped pendulum



Mathematical model $mLq'' + mg \sin q = 0$

Dividing by mL and writing $k = \frac{g}{L}$: $q'' + k \sin q = 0$

This is a nonlinear equation

When small value of $q \Rightarrow \sin q \sim q$ and approximate form $q'' + kq = 0$ has solution $A \cos \sqrt{kt} + B \sin \sqrt{kt}$

Critical points by linearization:

Set $q = y_1$ and $q' = y_2$ then nonlinear system

$$\begin{aligned} y_1' &= f_1(y_1, y_2) = y_2 \\ y_2' &= f_2(y_1, y_2) = -k \sin y_1 \end{aligned}$$

The right side both zero when $y_2 = 0$ and $y_1 = n\pi \Rightarrow$ infinity of critical points $(n\pi, 0)$ where $n = 0, \pm 1, \pm 2, \dots$

Consider $(0,0)$ the McLaurin series $\sin y_1 = y_1 - \frac{1}{6}y_1^3 + \dots \approx y_1$

Linearized system $\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix} \mathbf{y}$ and thus $\begin{aligned} y_1' &= y_2 \\ y_2' &= -ky_1 \end{aligned}$

We need $p = 0$, $q = \det \mathbf{A} = k (> 0)$ and $\Delta = p^2 - 4q = -4k$

From this we conclude that $(0,0)$ is a center always stable

Since $\sin q = \sin y_1$ is periodic with period 2π all critical points $(n\pi, 0)$ with $n = \pm 2, \pm 4, \dots$ are centers

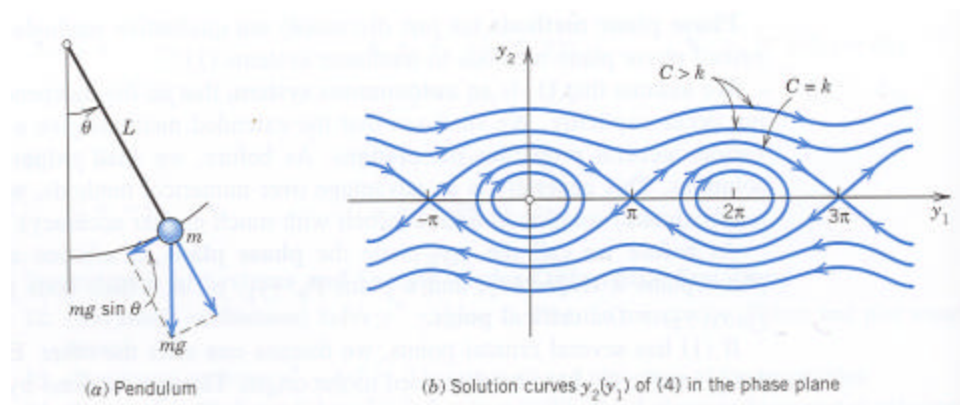
Consider now $(\mathbf{p}, 0) \Rightarrow$ setting $\mathbf{q} - \mathbf{p} = y_1$ and $(\mathbf{q} - \mathbf{p})' = \mathbf{q}' = y_2$ therefore

$$\sin \mathbf{q} = \sin(y_1 - \mathbf{p}) = -\sin y_1 = -y_1 + \frac{1}{6}y_1^3 - + \dots \approx -y_1$$

And linearized system $\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{pmatrix} 0 & 1 \\ k & 0 \end{pmatrix} \mathbf{y}$ and thus $\begin{matrix} y_1' = y_2 \\ y_2' = ky_1 \end{matrix}$

Which yields that $p = 0$, $q = \det \mathbf{A} = -k (< 0)$ and $\Delta = p^2 - 4q = 4k$

From which we conclude that $(n\pi, 0)$ with $n = \pm 1, \pm 3, \dots$ are saddle points which are always unstable



Example 2 –damped pendulum

Introduce damping term proportional to velocity

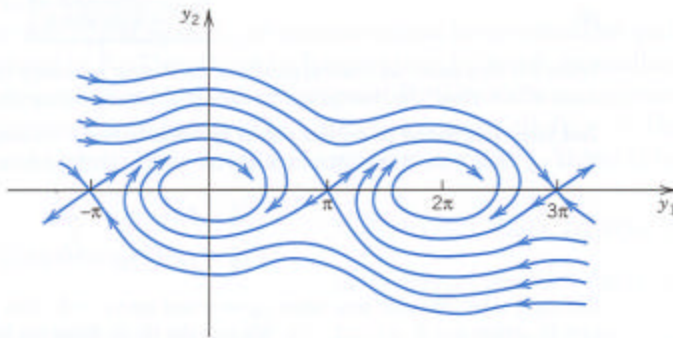
$$\mathbf{q}'' + c\mathbf{q}' + k \sin \mathbf{q} = 0$$

Linearized system
$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -k \sin y_1 - cy_2 \end{aligned}$$

Critical points at same locations $(0,0), (\pm p, 0), (\pm 2p, 0), \dots$

Consider $(0,0)$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{pmatrix} 0 & 1 \\ -k & -c \end{pmatrix} \mathbf{y} \quad \text{and} \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= -ky_1 - cy_2 \end{aligned}$$



For small damping spiral point

For critical point $(p, 0)$ we have $p = a_{11} + a_{22} = -c$, $q = \det \mathbf{A} = -k (< 0)$ and

$$\Delta = p^2 - 4q = c^2 + 4k$$

For $c > 0$ a saddle point

Damping = lost of energy – instead of closed trajectories we spiraling ones – no more trajectories connecting critical points

Transformation to a first-order equation in phase plane

Consider differential equation of second order $F(y, y', y'') = 0$

- Transform it to first order taking $y = y_1$ and $y' = y_2$ and transforming y'' by chain rule

$$y'' = y_2' = \frac{dy_2}{dt} = \frac{dy_2}{dy_1} \frac{dy_1}{dt} = \frac{dy_2}{dy_1} y_2$$

- Then $F\left(y, y_2, \frac{dy_2}{dy_1} y_2\right) = 0$

Example for free undamped pendulum

We get $\frac{dy_2}{dy_1} y_2 = -k \sin y_1$

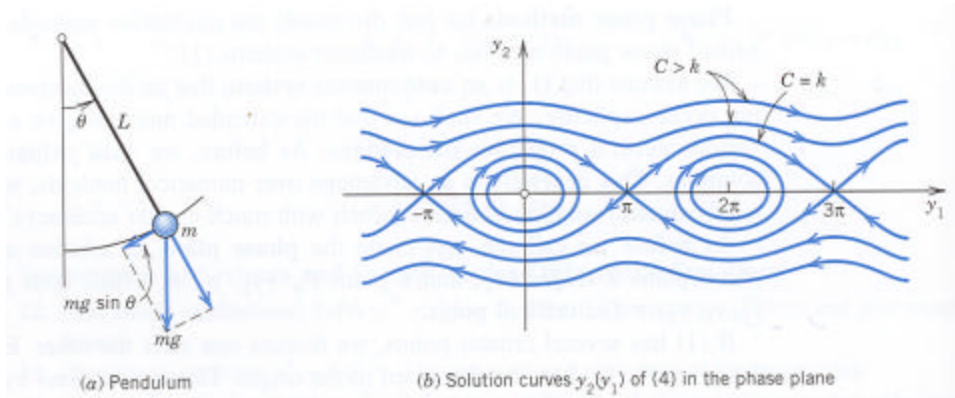
Separating the variables and integrating $\int y_2 dy_2 = \int -k \sin y_1 dy_1 + C$

$$\Rightarrow \frac{1}{2} y_2^2 = k \cos y_1 + C \text{ and multiplying by } mL^2 \Rightarrow \frac{1}{2} m (Ly_2)^2 - mL^2 k \cos y_1 = m \dot{E} C$$

These terms are energies

- y_2 is the angular velocity so Ly_2 is the velocity and first term is kinetic velocity
- Second term is the potential energy
- $m \dot{E} C$ is the total energy, which is constant (undamped system)

Type of motion depends on C



- Smallest possible $C = -k$ then $y_2 = 0$ and $\cos y_1 = 1$, the pendulum is at rest
- Change of direction of motion $y_2 = \mathbf{q}' = 0$ then $k \cos y_1 + C = 0$
- If $y_1 = \mathbf{p}$ then $\cos y_1 = -1$ and $C = k$, hence if $-k < C < k$ the pendulum oscillate (closed trajectories)
- If $C > k$ then $y_2 = 0$ is impossible and pendulum makes a whirly motion appearing a wavy trajectory
- If $C = k$ correspond to two separating trajectories connecting saddle point

Example 3 – van der Pol equation

There are physical systems such that for small oscillations energy is fed into the system (**negative damping**) while for large oscillation energy is taken from the system

We expect such system to approach a periodic behavior = closed trajectory = **limit cycle**

Van der Pol equation $y'' - m(1 - y^2) y' + y = 0$ with $m > 0$ a constant

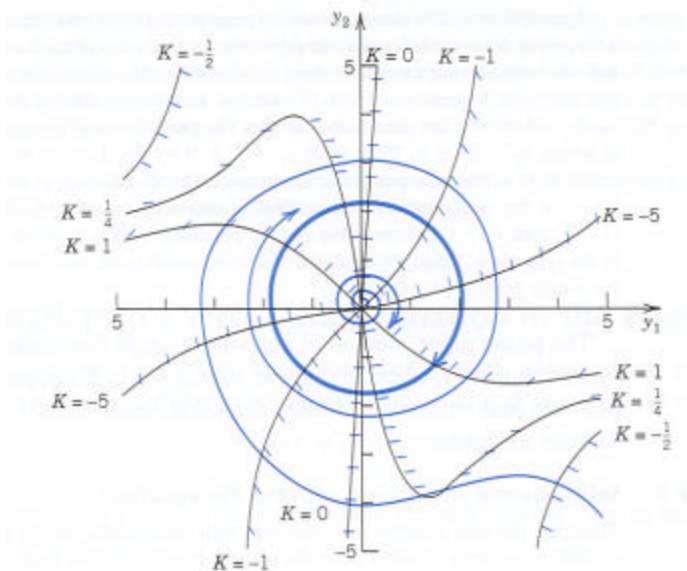
First occurred in study of electrical circuits containing vacuum tubes

The damping term $-m(1 - y^2) < 0$ for small oscillations $y^2 > 1$

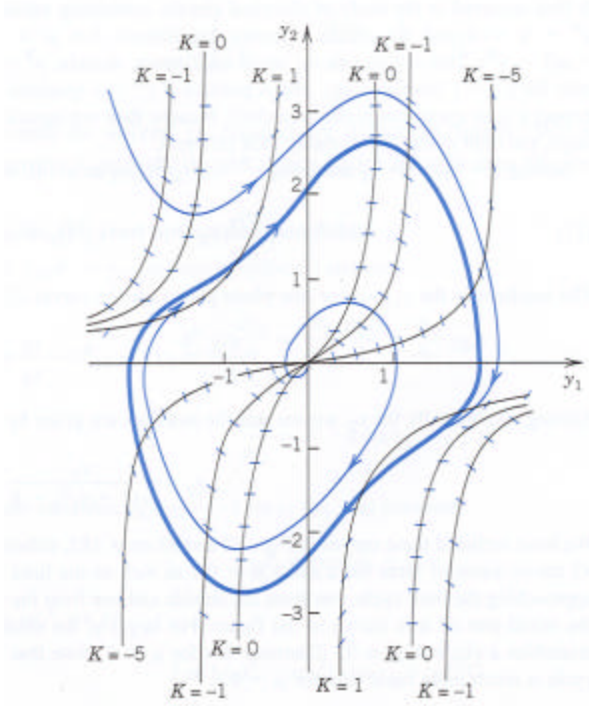
Setting $y = y_1$ and $y' = y_2 \Rightarrow \frac{dy_2}{dy_1} y_2 - m(1 - y_1^2) y_2 + y_1 = 0$

Isoclines in $y_1 y_2$ plane $\frac{dy_2}{dy_1} = K \Rightarrow \frac{dy_2}{dy_1} = m(1 - y_1^2) - \frac{y_1}{y_2} = K$

Solving $y_2 = \frac{y_1}{m(1 - y_1^2) - K}$ isoclines for $m = 0.1$



Limit cycle almost a circle because $y'' + y = 0$ and two trajectories approaching it from interior and exterior



For $m=1$ limit cycle no longer a circle and approach on trajectories more rapid

Non homogeneous linear systems